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Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities

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Abstract

We present new oscillation criteria for the second order forced ordinary differential equation with mixed nonlinearities:

$$\big(p(t)x'\big)' + q(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\alpha_i} \operatorname{sgn} x = e(t),$$

where $p(t),\ q(t),\ q_i(t),\ e(t) \in C[0,\infty)$, $p(t)$ is positive and differentiable, $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n$. No restriction is imposed on the forcing term $e(t)$ to be the second derivative of an oscillatory function. When $n = 1$, our results reduce to those of El-Sayed [M.A. El-Sayed, An oscillation criterion for a forced second order linear differential equation, Proc. Amer. Math. Soc. 118 (1993) 813–817], Wong [J.S.W. Wong, Oscillation criteria for a forced second linear differential equations, J. Math. Anal. Appl. 231 (1999) 235–240], Sun, Ou and Wong [Y.G. Sun, C.H. Ou, J.S.W. Wong, Interval oscillation theorems for a linear second order differential equation, Comput. Math. Appl. 48 (2004) 1693–1699] for the linear equation, Nazr [A.H. Nazr, Sufficient conditions for the oscillation of forced super-linear second order differential equations with oscillatory potential, Proc. Amer. Math. Soc. 126 (1998) 123–125] for the superlinear equation, and Sun and Wong [Y.G. Sun, J.S.W. Wong, Note on forced oscillation of nth-order sublinear differential equations, J. Math. Anal. Appl. 298 (2004) 114–119] for the sublinear equation.

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1. Introduction

We are here concerned with oscillatory behaviour of solutions of second order nonlinear differential equation

$$\big(p(t)x'\big)' + f(t,x) = 0, \quad t \geqslant 0, \tag{1}$$

where $p(t)$ is positive and differentiable and $f(t,x)$ is of the form

$$f(t,x) = q(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\alpha_i} \operatorname{sgn} x, \tag{2}$$

$\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$. As usual, a solution $x(t)$ of (1) is said to be oscillatory if it is defined on some ray $[T,\infty)$, $T \geqslant 0$, and has unbounded set of zeros. Equation (1) is said to be oscillatory if all solutions extendable throughout $[0,\infty)$ are oscillatory. When $f(t,x)$ takes the form (2), it is known that if $q(t)$, $q_i(t)$ are also positive and continuously differentiable then all solutions of (1) are extendable throughout $[0,\infty)$. However, when $q(t)$, $q_i(t)$ change signs as t tends to infinity, it is also known that Eq. (1) can have solutions with finite escape time, i.e. it can become infinite at some finite t. See Coffman and Wong [6]. Throughout this paper we shall, for simplicity, confine our discussion only to extendable solutions of Eq. (1).

When $p(t) \equiv 1$, $q(t) \equiv 0$, and $n = 1$ in (2), Eq. (1) becomes the familiar Emden–Fowler equation

$$x'' + q(t)|x|^{\alpha} \operatorname{sgn} x = 0, \quad \alpha > 0. \tag{3}$$

When $\alpha > 1$, Eq. (3) is known as the superlinear equation and when $0 < \alpha < 1$, it is known as the sublinear equation. The oscillation of Eq. (3) has been the subject of much attention during the last 50 years, see the seminal book by Agarwal, Grace and O'Regan [2]. Most of the results on oscillation of (3) are valid for the more general equation

$$x'' + q(t)f(x) = 0, \tag{4}$$

where $xf(x) > 0$ when $x \neq 0$ and $f(x)$ satisfies certain conditions of superlinearity and sublinearity, see [2]. In particular, $f(x)$ can be a finite sum of powers of x and if there exist in this sum exponents of x which are both greater than and less than 1, then Eq. (4) is known as Emden–Fowler equation of the mixed type. When $f(t,x)$ in (1) takes the form of (2) and Eq. (1) is of a mixed type, results on oscillation are more or less the same as when $q(t)$, $q_i(t)$, $i = 1,2,\ldots,n$, are non-negative. This is however not the case when $q(t)$, $q_i(t)$, $i = 1,2,\ldots,n$, are oscillatory. When $n = 1$, it is known (see Butler [4]) that Eq. (3) is oscillatory if $q(t)$ is periodic and of mean value zero, e.g. $\sin t$ or $\cos t$. When $f(x)$ is a finite sum of powers of x, the more general equation is also oscillatory as shown in a subsequent paper by Butler [5] see also [11]. However, for Eq. (1) with $f(t,x)$ given in the form of (2), it seems to us that no known oscillation criterion is applicable even to the simple equation of the mixed type:

$$x'' + (\sin t)x + (\cos t)|x|^{\alpha} \operatorname{sgn} x = 0, \quad \alpha \neq 1. \tag{5}$$

On the other hand, if we consider the forced equation

$$\big(p(t)x'\big)' + q(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\alpha_i} \operatorname{sgn} x = e(t), \tag{6}$$