Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 334 (2007) 799–803
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Weak topology and Browder–Kirk's theorem on hyperspace

Thakyin Hu [a], Jen-Chun Fang [b,*]

[a] *Department of Mathematics, Tamkang University, Taiwan*
[b] *Department of Electronic Engineering, Lunghwa University of Science and Technology, Taiwan*

Received 3 May 2006
Available online 11 January 2007
Submitted by B. Sims

Abstract

Let K be a weakly compact, convex subset of a Banach space X with normal structure. Browder–Kirk's theorem states that every non-expansive mapping T which maps K into K has a fixed point in K. Suppose now that $WCC(X)$ is the collection of all non-empty weakly compact convex subsets of X. We shall define a certain weak topology $\mathcal{T}_w$ on $WCC(X)$ and have the above-mentioned result extended to the hyperspace $(WCC(\boldsymbol{X}); \mathcal{T}_w)$.

Keywords: Fixed point; Non-expansive mapping; Normal structure; Hyperspace

1. Introduction

The Brouwer, Banach, and Schauder–Tychonoff fixed point theorems were published in the early 1900s. Due in part to the important applications of these results to various branches of mathematics, many mathematicians were attracted to the study of fixed point theory. Markov–Kakutani proved that every commutative family of continuous affine mappings of a compact convex set of a topological vector space into itself must have a common fixed point. Later in 1965 Browder and Kirk [4,10] proved that if K is a weakly compact, convex subset with normal structure of a Banach space X, then every non-expansive mapping $T: K \to K$ must have a fixed point. Recently, we obtained some results [6–9] which are closely related to the Markov–

* Corresponding author.
E-mail address: jcfang@mail.lhu.edu.tw (J.-C. Fang).

doi:10.1016/j.jmaa.2006.12.078

Kakutani theorem. It is the main purpose of this paper to extend Browder–Kirk's theorem to the hyperspace $WCC(\boldsymbol{X})$ where $WCC(\boldsymbol{X})$ is the collection of all non-empty weakly compact, convex subsets of the Banach space X.

2. Notations and preliminaries

Let K be a non-empty, bounded convex subset of a Banach space X with $\operatorname{diam}(K) = \sup\{d(x,y) = \|x-y\|\colon x, y \in K\}$. A point $x \in K$ is said to be a diametral point of K if $\sup\{d(x,y) = \|x-y\|\colon y \in K\} = \operatorname{diam}(K)$. K is said to have normal structure if for each convex set $M \subset K$, and M contains more than one point, then M has a non-diametral point (i.e. there exists $x \in M$ such that $\sup\{d(x,y) = \|x-y\|\colon y \in M\} < \operatorname{diam}(M)$). It is known that every compact, convex set has normal structure and also every convex subset of a uniformly convex Banach space has normal structure. Suppose now $BCC(X)$ is the collection of all non-empty, bounded, closed, convex subsets of X. For $A, B \in BCC(X)$, define $N(A;\varepsilon) = \{x \in X\colon d(x,a) = \|x-a\| < \varepsilon$ for some $a \in A\}$ and $h(A,B) = \inf\{\varepsilon > 0\colon A \subset N(B;\varepsilon)$ and $B \subset N(A;\varepsilon)\}$, equivalently $h(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$. Then h is known as the Hausdorff metric and the space $(BCC(X),h)$ is known as the hyperspace over X. Now let $WCC(X)$ be the collection of all non-empty weakly compact, convex subsets of X and $CC(\boldsymbol{X})$ be the collection of all non-empty compact convex subsets of X. For general X we have $CC(\boldsymbol{X}) \subsetneqq WCC(X) \subsetneqq BCC(X)$. The notion of weak convergence of bounded, closed, convex sets has been studied by many mathematicians [1–3,5,11]. Hu and Huang [8] defined a certain weak topology $\mathcal{T}_w$ on the hyperspace $CC(X)$ and they extended the classical Markov–Kakutani theorem to the hyperspace $(CC(X),\mathcal{T}_w)$. We shall now define a weak topology $\mathcal{T}_w$ on $WCC(X)$ and extend the Browder–Kirk's fixed point theorem to the hyperspace $(WCC(X),\mathcal{T}_w)$. Now let $\mathbf{Z}$ denote the complex plane, $CC(\mathbf{Z})$ the collection of all non-empty compact, convex subsets of $\mathbf{Z}$ and h the Hausdorff metric on $CC(\mathbf{Z})$. Suppose X^* is the topological dual of X and $x^* \in X^*$, it follows from the weak continuity and linearity of X^* that for each weakly compact, convex subset A of X, $x^*(A)$ is a compact convex subset of $\mathbf{Z}$ and hence x^* maps the space $WCC(X)$ into $CC(\mathbf{Z})$. We shall now prove the following lemma.

Lemma 1. *Suppose $A, B \in WCC(X)$, then $h(x^*(A), x^*(B)) \leqslant \|x^*\| h(A,B)$ for each $x^* \in X^*$. Thus $x^*\colon (WCC(X),h) \to (CC(\mathbf{Z}),h)$ is continuous (note that the same h is used to denote different Hausdorff metrics).*

Proof. Let $r > h(A,B)$. Then $A \subset N(B;r)$ and $B \subset N(A;r)$. Hence for each $a \in A$, there exists $b \in B$ such that $\|a-b\| < r$ and consequently $\|x^*(a) - x^*(b)\| = \|x^*(a-b)\| \leqslant \|x^*\|\,\|a-b\| < \|x^*\| r$, which in turn implies that $x^*(A) \subset N(x^*(B), \|x^*\| r)$. Similarly $x^*(B) \subset N(x^*(A), \|x^*\| r)$. Hence $h(x^*(A), x^*(B)) \leqslant \|x^*\| r$, which implies that $h(x^*(A), x^*(B)) \leqslant \|x^*\| h(A,B)$ and the proof is complete. □

We shall need the following Lemma 2 which is easily verifiable and is stated without proof.

Lemma 2. *Let $A, B, C, D \in WCC(X)$. Then*

(a) *$A = B$ if and only if $x^*(A) = x^*(B)$ for each $x^* \in X^*$,*
(b) *$h(A+B, C+D) \leqslant h(A,C) + h(B,D)$.*