Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 334 (2007) 834–846
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A formulation of Noether's theorem for fractional problems of the calculus of variations

Gastão S.F. Frederico, Delfim F.M. Torres *

Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

Received 3 February 2006
Available online 16 January 2007
Submitted by H.M. Srivastava

Abstract

Fractional (or non-integer) differentiation is an important concept both from theoretical and applicational points of view. The study of problems of the calculus of variations with fractional derivatives is a rather recent subject, the main result being the fractional necessary optimality condition of Euler–Lagrange obtained in 2002. Here we use the notion of Euler–Lagrange fractional extremal to prove a Noether-type theorem. For that we propose a generalization of the classical concept of conservation law, introducing an appropriate fractional operator.

Keywords: Calculus of variations; Fractional derivatives; Noether's theorem

1. Introduction

The notion of conservation law—first integral of the Euler–Lagrange equations—is well known in Physics. One of the most important conservation laws is the integral of energy, discovered by Leonhard Euler in 1744: when a Lagrangian $L(q,\dot{q})$ corresponds to a system of conservative points, then

$$-L(q,\dot{q}) + \partial_2 L(q,\dot{q}) \cdot \dot{q} \equiv \text{constant} \tag{1}$$

* Corresponding author.
E-mail addresses: gfrederico@mat.ua.pt (G.S.F. Frederico), delfim@mat.ua.pt (D.F.M. Torres).
URL: http://www.mat.ua.pt/delfim (D.F.M. Torres).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.01.013

holds along all the solutions of the Euler–Lagrange equations (along the extremals of the autonomous variational problem), where $\partial_2 L(\cdot,\cdot)$ denote the partial derivative of function $L(\cdot,\cdot)$ with respect to its second argument. Many other examples appear in modern physics: in classic, quantum, and optical mechanics; in the theory of relativity, etc. For instance, in classic mechanics, beside the conservation of energy (1), it may occur conservation of momentum or angular momentum. These conservation laws are very important: they can be used to reduce the order of the Euler–Lagrange differential equations, thus simplifying the resolution of the problems.

In 1918 Emmy Noether proved a general theorem of the calculus of variations, that permits to obtain, from the existence of variational symmetries, all the conservation laws that appear in applications. In the last decades, Noether's principle has been formulated in various contexts (see [16,17] and references therein). In this work we generalize Noether's theorem for problems having fractional derivatives.

Fractional differentiation plays nowadays an important role in various seemingly diverse and widespread fields of science and engineering: physics (classic and quantum mechanics, thermodynamics, optics, etc.), chemistry, biology, economics, geology, astrophysics, probability and statistics, signal and image processing, dynamics of earthquakes, control theory, and so on [3,6,8,9]. Its origin goes back more than 300 years, when in 1695 L'Hopital asked Leibniz the meaning of $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$. After that, many famous mathematicians, like J. Fourier, N.H. Abel, J. Liouville, B. Riemann, among others, contributed to the development of the fractional calculus [6,10,14].

F. Riewe [12,13] obtained a version of the Euler–Lagrange equations for problems of the calculus of variations with fractional derivatives, that combines the conservative and non-conservative cases. More recently, O. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann–Liouville sense [1]. Then these Euler–Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [4]. Here we use the results of [1] to generalize Noether's theorem for the more general context of the fractional calculus of variations.

The paper is organized in the following way. In Section 2 we recall the notions of right and left Riemann–Liouville fractional derivatives, that are needed for formulating the fractional problem of the calculus of variations. There are many different ways to approach classical Noether's theorem. In Section 3 we review the only proof that we are able to extend, with success, to the fractional context. The method is based on a two-step procedure: it starts with an invariance notion of the integral functional under a one-parameter infinitesimal group of transformations, without changing the time variable; then it proceeds with a time-reparameterization to obtain Noether's theorem in general form. The intended fractional Noether's theorem is formulated and proved in Section 4. Two illustrative examples of application of our main result are given in Section 5. We finish with Section 6 of conclusions and some open questions.

2. Riemann–Liouville fractional derivatives

In this section we collect the definitions of right and left Riemann–Liouville fractional derivatives and their main properties [1,10,14].

Definition 1 *(Riemann–Liouville fractional derivatives).* Let f be a continuous and integrable function in the interval $[a,b]$. For all $t \in [a,b]$, the left Riemann–Liouville fractional derivative ${}_aD_t^\alpha f(t)$, and the right Riemann–Liouville fractional derivative ${}_tD_b^\alpha f(t)$, of order α, are defined in the following way: