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On existence of degenerate circle-preserving maps ☆

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Abstract

Recently, B. Li and Y. Wang proved that if $f:\mathbb{R}^n_\infty \to \mathbb{R}^n_\infty$ $(n \geqslant 2)$ is a circle-preserving map, then f is a Möbius transformation if and only if f is a non-degenerate map, where a map f is degenerate if the image $f(\mathbb{R}^n_\infty)$ is a circle. Furthermore, they conjectured that there should exist no degenerate map, or equivalently, f is a Möbius transformation if and only if f is a circle-preserving map. In this note, we construct a degenerate circle-preserving map to show that the conjecture is not true.

Keywords: Möbius transformation; Circle-preserving map; Degenerate map

1. Introduction

As usual, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let $\mathbb{R}^n_\infty = \mathbb{R}^n \cup \{\infty\}$. We write $\mathbb{H}^n$ for the n-dimensional hyperbolic space and restrict that a geodesic in $\mathbb{H}^n$ has two endpoints on the ideal boundary $\partial\mathbb{H}^n$. In the sequel, we prescribe $n \geqslant 2$.

A map f of $\mathbb{R}^n_\infty$ $(\mathbb{R}^n, \mathbb{H}^n)$ into itself is called circle-preserving (line-preserving, geodesic-preserving) if the image of any circle (line, geodesic) under f is still a circle (line, geodesic). Moreover, the map f is called degenerate if its image $f(\mathbb{R}^n_\infty)$ $(f(\mathbb{R}^n), f(\mathbb{H}^n))$ is a circle (line, geodesic), and otherwise, f is called non-degenerate.

A Möbius transformation acting on $\mathbb{R}^n_\infty$ is obviously circle-preserving. The situation about the converse intrigues many authors and some interesting results (cf. [1–3,5]) have been ob-

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tained. In a very recent article [4], Li and Wang refined the conditions for a circle-preserving (line-preserving, geodesic-preserving) map to be a(n) Möbius (affine, isometric) transformation, that is,

Theorem A. *Suppose that* $f:\mathbb{R}^n_\infty \to \mathbb{R}^n_\infty$ *is a circle-preserving map. Then* f *is a Möbius transformation if and only if* f *is non-degenerate.*

Theorem B. *Suppose that* $f:\mathbb{R}^n \to \mathbb{R}^n$ *is a line-preserving map. Then* f *is an affine transformation if and only if* f *is non-degenerate.*

Theorem C. *Suppose that* $f:\mathbb{H}^n \to \mathbb{H}^n$ *is a geodesic-preserving map. Then* f *is a hyperbolic isometry if and only if* f *is non-degenerate.*

A natural problem arises from Theorems A, B and C, i.e., does there exist a degenerate circle-preserving (line-preserving, geodesic-preserving) map? For the latter two cases in $\mathbb{R}^n$ and $\mathbb{H}^n$, Li and Wang constructed certain degenerate line-preserving and geodesic-preserving maps, respectively. Meanwhile, they believed that there should exist no degenerate circle-preserving map on $\mathbb{R}^n_\infty$. More precisely, they proposed

Conjecture. *Degenerate circle-preserving maps do not exist. Equivalently,* f *is a Möbius transformation if and only if* f *is a circle-preserving map.*

The main aim of this paper is to construct a degenerate circle-preserving map to show that the above conjecture is not true. Since for a degenerate circle-preserving map f, the inverse images of every point in $f(\mathbb{R}^n_\infty)$ are dense in $\mathbb{R}^n_\infty$, f is necessarily nowhere continuous. However, certain real-analytic degenerate line-preserving (geodesic-preserving) maps exist! We will also give such examples since the degenerate maps given in [4] are not continuous at some points.

2. Construction of degenerate maps

Construction of degenerate circle-preserving map: We divide the construction of the required map into four steps.

Step 1. Define a map $g_1:\mathbb{R}^n_\infty \to \mathbb{R}^1$ as follows:

$$g_1(x) = \begin{cases} \sum_{i=1}^n |x_i|, & x \in \mathbb{R}^n, \\ 0, & x = \infty, \end{cases}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_\infty$.

Step 2. Define an equivalence relation on $\mathbb{R}$: x and y in $\mathbb{R}$ are equivalent if and only if $x - y \in \mathbb{Q}$. Given x in $\mathbb{R}$, let $\tilde{x}$ denote the set of all elements $y \in \mathbb{R}$ equivalent to x. We denote by $\mathbb{R}_\mathbb{Q}$ the set of all equivalence classes $\tilde{x}$. We claim that the cardinality card($\mathbb{R}_\mathbb{Q}$) of $\mathbb{R}_\mathbb{Q}$ is $\aleph$ instead of $\aleph_0$. Suppose to the contrary, i.e., card($\mathbb{R}_\mathbb{Q}$) is $\aleph_0$; in other words, $\mathbb{R}_\mathbb{Q}$ is a countable set. Choose one representative element from every equivalence class $\tilde{x}$ in $\mathbb{R}_\mathbb{Q}$ to form a subset W in $\mathbb{R}$. Whence, W is countable. On the other hand, we can assume that the rational numbers set $\mathbb{Q} = \{r_1, r_2, \ldots, r_k, \ldots\}$. Notice that

$$\mathbb{R}^1 = \bigcup_{k=1}^{\infty} \big(W + \{r_k\}\big).$$