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Metrizability of spaces of holomorphic functions with the Nachbin topology ☆

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Abstract

In this paper we prove, among other things, that the space of all holomorphic functions on an open subset U of a complex metrizable space E, endowed with the Nachbin ported topology, is metrizable only if E has finite dimension. This answers a question by Mujica in [J. Mujica, Gérmenes holomorfos y funciones holomorfas en espacios de Fréchet, Publicaciones del Departamento de Teoría de Funciones, Universidad de Santiago, Spain, 1978].

Keywords: Holomorphic function; Nachbin topology; Metrizable space

1. Introduction

Let us consider an open subset U of $\mathbb{C}^n$, $n = 1, 2, \ldots$, then the space $\mathcal{H}(U)$ of all holomorphic functions on U endowed with the compact open topology τ_o enjoys very interesting properties from the functional analysis point of view. For instance, it is a Fréchet space, and hence it is bornological and barrelled.

Let E be a complex locally convex space and let U be an open subset of E. A mapping $f : U \to \mathbb{C}$ is said to be holomorphic in U if it is continuous and for every $z \in U$ and $w \in E$ the mapping

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$$\lambda \mapsto f(z + \lambda w)$$

is holomorphic in a neighborhood of 0 in $\mathbb{C}$. Holomorphic mappings on open subsets of locally convex spaces have Taylor series expansions and many other interesting properties (see [4]).

On the space $\mathcal{H}(U)$ of all holomorphic functions on U we can also consider the compact open topology τ_o defined by the seminorms

$$f \in \mathcal{H}(U) \mapsto |f|_K := \sup\{|f(z)|: z \in K\}$$

where K ranges over the family of all compact subsets of U. This topology, in the infinite dimensional case, does not share, in general, the good properties it has in the finite dimensional case. For this reason other natural topologies, finer than τ_o, are often considered on $\mathcal{H}(U)$. Here we are going to consider just one, the well-known Nachbin ported topology τ_ω, defined by the seminorms ported by the compact subsets of U; we recall that a seminorm p on $\mathcal{H}(U)$ is *ported* by the compact subset K of U if for every open neighborhood V of K in U there is a constant $C > 0$ such that

$$p(f) \leqslant C|f|_V \quad \text{for all } f \in \mathcal{H}(U).$$

We first prove that when E is metrizable and $(\mathcal{H}(U), \tau_\omega)$ is also metrizable, then E is normed. In the case of Banach spaces it is known that the metrizability of $(\mathcal{H}(U), \tau)$, for $\tau = \tau_o$ or τ_ω, implies that E has finite dimension. The proof of that can be seen in [2, 16.10], it relies deeply on the completeness of the Banach spaces and on results by Josefson [6] and Aron [1]. It cannot be applied to the general case of normed spaces.

Here we prove the non-metrizability of $(\mathcal{H}(U), \tau_\omega)$ for every open subset U of every infinite dimensional metrizable (not necessarily complete) space E. On the other hand, it is known that if E has finite dimension then $(\mathcal{H}(U), \tau_\omega)$ is metrizable for every open U of E, because in such a case $\tau_\omega = \tau_o$ on $\mathcal{H}(U)$.

2. The results

We start with a proposition which, for the study of the metrizability of $(\mathcal{H}(U), \tau_\omega)$, reduces the case of a metrizable space to the normed one.

Proposition 1. *If E is a metrizable space and for some locally convex topology τ, with $\tau_o \leqslant \tau \leqslant \tau_\omega$, the space (E', τ) is also metrizable, then E is a normed space.*

Proof. Let τ be as in the proposition, and let $\{V_k\}_{k=1}^\infty$ be a decreasing fundamental sequence of absolutely convex closed neighborhoods of 0 in E. Then each V_k° is τ-bounded. Indeed, since the τ_ω continuous seminorms on E' are ported by $\{0\}$, if p is a τ_ω continuous seminorm on E', there is a constant $C_k > 0$ such that

$$p(\varphi) \leqslant C_k|\varphi|_{V_k} \quad \text{for all } \varphi \in E',$$

and so

$$p(\varphi) \leqslant C_k \quad \text{for all } \varphi \in V_k^\circ.$$

Moreover $V_1^\circ \subset V_2^\circ \subset \cdots$ is a fundamental sequence of bounded subsets for (E', τ). Indeed, every τ bounded set is τ_o bounded and, since E is metrizable, the τ_o bounded subsets are locally