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On the order of starlikeness of the shifted Gauss hypergeometric function

Reinhold Küstner

Universität Kassel, FB 17 Mathematik/Informatik, AG Computational Mathematics, Heinrich-Plett-Straße 40, 34132 Kassel, Germany

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Abstract

We determine for several ranges of real parameters the order of starlikeness of the shifted Gauss hypergeometric function and we give some consequences of our results, in particular some mapping properties of the Carlson–Shaffer convolution operator.

Keywords: Gauss hypergeometric function; Univalence; Order of starlikeness; Order of convexity; Zerofreeness in the unit disk; Carlson–Shaffer convolution operator

1. Introduction and statement of results

Let $\mathbb{D} := \{z \in \mathbb{C}\colon\ |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathcal{H}$ denote the set of all functions which are analytic in $\mathbb{D}$. For a function $f \in \mathcal{H}$ with $f(0) = 0 \neq f'(0)$ its order of starlikeness (with respect to zero) is defined by

$$\sigma(f) := \inf_{z\in\mathbb{D}} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \in [-\infty, 1],$$

and if at least $f'(0) \neq 0$ then the order of convexity of f is defined by

$$\kappa(f) := \sigma(zf') = 1 + \inf_{z\in\mathbb{D}} \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \in [-\infty, 1].$$

E-mail address: kuestner@mathematik.uni-kassel.de.

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As is well known, f is starlike, i.e. $\sigma(f) \geqslant 0$, if and only if f is univalent in $\mathbb{D}$ (i.e. one-to-one) with $f(\mathbb{D})$ being starlike with respect to zero; and f is convex, i.e. $\kappa(f) \geqslant 0$, if and only if f is univalent in $\mathbb{D}$ with $f(\mathbb{D})$ being convex. It is further known that if $\kappa(f) \geqslant -1/2$ then f is univalent in $\mathbb{D}$ with $f(\mathbb{D})$ being convex in (at least) one direction, see [12,21] and [16, Theorem 2.24, pp. 71, 73].

If $\sigma(f) > -\infty$ then $z \mapsto zf'(z)/f(z)$ can have no pole in $\mathbb{D}$, i.e. $z \mapsto f(z)/z$ has no zero in $\mathbb{D}$. For this reason we make the convention that $\sigma(f) := -\infty$ only if $z \mapsto f(z)/z$ has no zero in $\mathbb{D}$ and $\mathrm{Re}(zf'(z)/f(z))$ is not bounded from below in $\mathbb{D}$, whereas $\sigma(f)$ is considered to be not defined if $z \mapsto f(z)/z$ has a zero in $\mathbb{D}$. And, of course, we shall also make the corresponding convention for $\kappa(f)$, i.e. $\kappa(f) := -\infty$ only if f' has no zero in $\mathbb{D}$ and $\mathrm{Re}(zf''(z)/f'(z))$ is not bounded from below in $\mathbb{D}$, whereas $\kappa(f)$ is considered to be not defined if f' has a zero in $\mathbb{D}$.

In this paper we determine for several ranges of real parameters a, b, c the order of starlikeness of the shifted Gauss hypergeometric function $z \mapsto zF(a,b,c,z)$. Here the Gauss hypergeometric function $z \mapsto F(a,b,c,z)$ depends on the three parameters $a, b, c \in \mathbb{C}$, $-c \notin \mathbb{N} := \{0, 1, 2, \ldots\}$, and is defined for $z \in \mathbb{D}$ by

$$F(a,b,c,z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{1}$$

where $(a)_n$ is the Pochhammer symbol, i.e. $(a)_0 := 1$ and $(a)_{n+1} := (a)_n (a+n)$ for all $n \in \mathbb{N}$, so that in particular $(1)_n = n!$ and $F(a,b,c,z) = 1$ if $abz = 0$.

Our results will also yield the zerofreeness in $\mathbb{D}$ as well as the order of convexity of the hypergeometric function, and if $b = 1$ or if $c = 2$ then they further yield the order of convexity of the shifted hypergeometric function.

As another consequence of our results on the order of starlikeness of the shifted hypergeometric function, we shall also obtain three mapping properties of the Carlson–Shaffer convolution operator on classes of starlike functions of order α and two results on the convolution of starlike or convex functions of order α.

1.1. Results on the order of starlikeness of the shifted hypergeometric function

In what follows we summarize known as well as new results on the order of starlikeness $\sigma(zF(a,b,c,z))$ of the shifted hypergeometric function $z \mapsto zF(a,b,c,z)$.

Theorem 1.

(a) [8, Theorem 1.1, Remark 2.3] *If* $0 < a \leqslant b \leqslant c$ *then*

$$1 - \frac{ab}{c+b} \leqslant \sigma\big(zF(a,b,c,z)\big) = 1 - \frac{F'(a,b,c,-1)}{F(a,b,c,-1)} \leqslant 1 - \frac{ab}{2c}.$$

(b) [8, Remark 1.2] *If* $-1 \leqslant a < 0 < b \leqslant c \leqslant a+b+1$ *then*

$$\sigma\big(zF(a,b,c,z)\big) = 1 + \frac{F'(a,b,c,1)}{F(a,b,c,1)} = -\infty.$$

(c) [19, Theorem 2], [8, Remark 1.2] *If* $-1 \leqslant a < 0 < b \leqslant a+b+1 < c$ *then*

$$\sigma\big(zF(a,b,c,z)\big) = 1 + \frac{F'(a,b,c,1)}{F(a,b,c,1)} = 1 + \frac{ab}{c-a-b-1}.$$