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Fixed points of generalized e-concave (generalized e-convex) operators and their applications ☆

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Abstract

In this paper, we present the definitions of generalized e-concave operators and generalized e-convex operators, which are the generalizations of e-concave operators and e-convex operators, respectively. Without compactness or continuity assumption of generalized e-concave operators and generalized e-convex operators, we have proved the existence, uniqueness and monotone iterative techniques of their fixed points. Our results are even new to e-concave operators and e-convex operators. Finally, we apply the results to the singular boundary value problems for second order differential equations.

Keywords: Generalized e-concave operator; Generalized e-convex operator; Fixed point; Two point boundary value problem; Increasing operator; Decreasing operator

1. Introduction

[1] states the definitions and properties of e-concave operators and e-convex operators which have been investigated by Guo and Lakshmikantham in [2]. More research can be found in [3]. In this paper, we present the definitions of generalized e-concave operators and generalized e-convex operators, which are the generalizations of e-concave operators and e-convex operators, respectively. Without compactness or continuity assumption of generalized e-concave operators

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and generalized e-convex operators, we have proved the existence, uniqueness and monotone iterative techniques of their fixed points. As corollaries, we also obtain the existence of fixed points of e-concave operators and e-convex operators. Finally, we apply the results to the singular boundary value problems for second order differential equations, which improved the previous results.

Let E be a real Banach space, P be a cone of E and "$\leqslant$" be the partial ordering defined by P, $e \in P - \{\theta\}$ and

$$C_e = \{x \in E \mid \text{there exist positive numbers } \alpha, \beta \text{ such that } \alpha e \leqslant x \leqslant \beta e\}.$$

Set

$$E_e = \{x \in E \mid \text{there exists } \lambda > 0 \text{ such that } -\lambda e \leqslant x \leqslant \lambda e\},$$

and

$$\|x\|_e = \inf\{\lambda > 0 \mid -\lambda e \leqslant x \leqslant \lambda e\}, \quad \forall x \in E_e.$$

It is easy to see that E_e becomes a normed linear space under the norm $\|.\|_e$. $\|.\|_e$ is called the e-norm of the element $x \in E_e$.

Recall that cone P is said to be normal if there exists a positive constant N such that $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$, the smallest N is called the normal constant of P.

Definition 1.1. Let $A : P \to P$ be an operator and $e > \theta$. Suppose that

(i) $Ae \in C_e$,
(ii) there exists a real number $\eta = \eta(x,t) > 0$ such that

$$A(tx) \geqslant t(1+\eta)Ax, \quad \forall x \in C_e,\ 0 < t < 1. \tag{1.1}$$

Then A is called a generalized e-concave operator.

Similarly, if in the above definition, we replace (ii) by the following

$$\text{(ii)}' \quad A(tx) \leqslant \big(t(1+\eta)\big)^{-1} Ax, \quad \forall x \in C_e,\ 0 < t < 1, \tag{1.2}$$

then A is called a generalized e-convex operator.

Remark 1.1. (1.1) implies

$$A(\lambda x) \leqslant \lambda\left[1 + \eta\left(\lambda x, \frac{1}{\lambda}\right)\right]^{-1} Ax, \quad \forall x \in C_e,\ \lambda > 1; \tag{1.3}$$

and (1.2) implies

$$A(\lambda x) \geqslant \lambda^{-1}\left[1 + \eta\left(\lambda x, \frac{1}{\lambda}\right)\right] Ax, \quad \forall x \in C_e,\ \lambda > 1. \tag{1.4}$$

Conversely, (1.3) implies (1.1) and (1.4) implies (1.2).

Remark 1.2. If we replace the condition (i) in Definition 1.1 by the following

$$\text{(i)}' \quad \text{for any } x \in P - \{\theta\}, \quad Ax \in C_e,$$

then A is called an e-concave operator. Obviously, an e-concave operator is a generalized e-concave operator.