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Reduction and transformation formulae for bivariate basic hypergeometric series

C.-Z. Jia *, T.-M. Wang

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China

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Abstract

The main object of this paper is to establish several bivariate basic hypergeometric series identities by means of elementary series manipulation. Some of them can be applied to yield transformation and reduction formulae for q-Kampé de Fériet functions.

Keywords: Basic hypergeometric series; Kampé de Fériet functions; Jackson's transformation; q-Pfaff–Saalschütz theorem; Sears–Carlitz transformation; Watson's q-Whipple transformation

1. Introduction

For two indeterminates x and q, the shifted factorial is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}\left(1-q^k x\right) \quad \text{with } n = 1, 2, \ldots.$$

When $|q| < 1$, we have the following well-defined infinite product expressions:

$$(x;q)_\infty = \prod_{k=0}^{\infty}\left(1-q^k x\right) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} \quad \text{for } n \in \mathbb{Z}.$$

* Corresponding author.
E-mail address: cangzhijia@yahoo.com.cn (C.-Z. Jia).

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For the sake of brevity, we also write the factorial product compactly as

$$[a, b, \ldots, c; q]_n := (a; q)_n (b; q)_n \cdots (c; q)_n.$$

Following Gasper and Rahman [5], the basic hypergeometric series is defined by

$${}_{1+r}\Phi_s \left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \frac{[a_0, a_1, \ldots, a_r; q]_n}{[q, b_1, \ldots, b_s; q]_n} z^n \tag{1.1}$$

provided that no zero factors appear in the denominator on the right-hand side, i.e., none of the denominator parameters $\{b_k\}_{k=1}^{s}$ has the form q^{-m} with $m \in \mathbb{N}_0$.

As the q-analogue of Kampé de Fériet function, Srivastava and Karlsson [8, p. 349] defined the generalized bivariate basic hypergeometric function by

$$\Phi_{\mu:u;v}^{\lambda:r;s} \left[\begin{matrix} \alpha_1, \ldots, \alpha_\lambda\colon & a_1, \ldots, a_r; & c_1, \ldots, c_s; & q\colon x, y \\ \beta_1, \ldots, \beta_\mu\colon & b_1, \ldots, b_u; & d_1, \ldots, d_v; & i, j, k \end{matrix} \right] \tag{1.2a}$$

$$= \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \ldots, \alpha_\lambda; q]_{m+n}}{[\beta_1, \ldots, \beta_\mu; q]_{m+n}} \frac{[a_1, \ldots, a_r; q]_m [c_1, \ldots, c_s; q]_n}{[b_1, \ldots, b_u; q]_m [d_1, \ldots, d_v; q]_n} \frac{x^m y^n q^{i\binom{m}{2} + j\binom{n}{2} + kmn}}{(q; q)_m (q; q)_n}. \tag{1.2b}$$

It is not hard to check that when $i, j, k \in \mathbb{N}_0$, the double series $\Phi_{\mu:u;v}^{\lambda:r;s}$ is absolutely convergent for $|x| < 1$, $|y| < 1$ and $|q| < 1$.

In [1], Buschman and Srivastava gave some double hypergeometric series identities with an arbitrary parameter. The corresponding q-analogues were studied by Karlsson [6] and Srivastava and Jain [7]. Recently, Chan et al. [2], Chen and Srivastava [3] and Chu and Srivastava [4] also obtained several double series identities by using quadratic transformations and summation theorems. Motivated by the aforementioned works, this paper will further investigate bivariate q-series and prove several reduction and transformation formulae for q-Kampé de Fériet functions. Some of these q-series identities may be considered as q-analogues of the results that appeared in [2, Eq. (13)], [3, Eqs. (2.1), (2.10), (3.8)] respectively.

The proofs of the theorems all involve, first, a series rearrangement of the form $\sum_{m,n=0}^{\infty} \cdots = \sum_{k=0}^{\infty} \sum_{m}^{k} \cdots$. The opposite step, where we go back to $\sum_{m,n=0}^{\infty} \cdots$, also occurs.

2. Reduction formulae and double series identities

By means of series rearrangements, we now prove two general reduction formulae for double q-series, which will then be applied to derive several double series summation identities.

Theorem 1 *(Reduction formula). For an arbitrary complex sequence $\{\Omega(n)\}_{n=0}^{\infty}$, there holds the following transformation*

$$\sum_{m,n=0}^{\infty} \Omega(m+n) \frac{[a, qa^{\frac{1}{2}}, b; q]_m}{[q, a^{\frac{1}{2}}, qa/b; q]_m} \frac{(1/b^2; q)_n}{(q; q)_n} b^{-2m} x^{m+n} \tag{2.1a}$$

$$= \sum_{n=0}^{\infty} \Omega(n) \frac{(a/b^2; q)_n (1/b; q)_n (-qa^{\frac{1}{2}}/b; q)_n}{(q; q)_n (qa/b; q)_n (-a^{\frac{1}{2}}/b; q)_n} x^n \tag{2.1b}$$

provided that both series displayed above are absolutely convergent.