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On several results about convex set functions

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Abstract

In 1979, in an interesting paper, R.J. Morris introduced the notion of convex set function defined on an atomless finite measure space. After a short period this notion, as well as generalizations of it, began to be studied in several papers. The aim was to obtain results similar to those known for usual convex (or generalized convex) functions. Unfortunately several notions are ambiguous and the arguments used in the proofs of several results are not clear or not correct. In this way there were stated even false results. The aim of this paper is to point out that using some simple ideas it is possible, on one hand, to deduce the correct results by means of convex analysis and, on the other hand, to emphasize the reasons for which there are problems with other results.

Keywords: Atomless measure space; Conjugate; Lsc convex hull; M-convex function; M-convex set; Subdifferential

1. Introduction

Convex analysis is a well established and nice mathematical discipline with important applications. Many results concern conjugate and subdifferential calculus, as well as optimality conditions in convex programming. In this context the classical Fenchel duality formula

$$\inf_{x\in X}\big[f(x)+g(x)\big]=\max_{u^*\in X^*}\big[-f^*\big(u^*\big)-g^*\big(-u^*\big)\big], \tag{1}$$

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established initially for proper convex functions f and g such that f (or g) is continuous at a point in the intersection of their domains, plays a central role. Replacing f by $f - x^*$ with $x^* \in X^*$, we observe that the above formula becomes

$$(f+g)^*(x^*) = \min_{u^* \in X^*} \big[f^*(u^*) + g^*(x^* - u^*) \big]; \tag{2}$$

in turn, for $x^* = 0$, this formula yields (1). As pointed out by Hiriart-Urruty in [6], when for the proper convex functions f and g formula (2) holds for every $x^* \in X^*$, one has

$$\partial(f+g)(x) = \partial f(x) + \partial g(x) \quad \forall x \in X.$$

Obtaining formula (1) without convexity of f and g attracted several mathematicians. Because $f \geqslant \operatorname{co} f$ and $f^* = (\operatorname{co} f)^*$, we have

$$\begin{aligned} \inf(f+g) &\geqslant \inf(\operatorname{co} f + \operatorname{co} g) \geqslant \sup\big\{ -(\operatorname{co} f)^*(u^*) - (\operatorname{co} g)^*(-u^*) \bigm| u^* \in X^* \big\} \\ &= \sup\big\{ -f^*(u^*) - g^*(-u^*) \bigm| u^* \in X^* \big\} \end{aligned}$$

(in fact one can replace co by $\overline{\text{co}}$). This shows that in order to prove (1) for f and g, one must prove it for the convex functions $\operatorname{co} f$ and $\operatorname{co} g$ (or even $\overline{\text{co}}\, f$ and $\overline{\text{co}}\, g$) using general results from convex analysis and to see if $f + g$ and $\operatorname{co} f + \operatorname{co} g$ have the same infimum.

The majority of the mathematicians dealing with nonconvex functions prove such results using separation theorems (as for the convex case) instead of reducing them to the convex case (see, e.g., [4,14] for functions defined on a general linear space). A similar situation happens for convex set functions, that is extended real-valued functions defined on the class of measurable subsets of an atomless finite measure space $(E, \mathcal{A}, \mu)$ satisfying a supplementary (convexity) condition. Such a function has the particularity, when identified with a function defined on $L_\infty(E, \mathcal{A}, \mu)$, that the weak* closure of its epigraph is convex. The first paper on convex set functions is the interesting one of Morris [13]. After that several papers dealt not only with convex set functions but also with generalized convexity notions for set functions. Unfortunately in several papers on this topic the notions are ambiguous and the proofs of several statements are not correct (at least not convincing), but some of them are even false. Our aim in this note is to point out those statements for which we have doubts and to provide counterexamples for some of them. Our main motivation in doing this is provided by the recent papers [1] (a survey one) and [12] where the majority of problematic statements are cited and used.

In view of these difficulties we devote Section 2 to detailed preliminaries making clear notions related to extensions of functions. In Section 3 we state some results concerning the conjugate and subdifferential of the function $f_S := f + \iota_S$ with f a lower semicontinuous and convex function and S an arbitrary set, with a special emphasize of functions defined on L_∞. Section 4 deals with Morris convex set functions. Because in the literature there were given several variants for the convexity of an extended real-valued convex set function, we fix the definition and state some results on the conjugate and subdifferential of M-convex functions. In this way we recover some known results and point out some false results from the literature. In Section 5 we discuss the Gâteaux differentiability of two set functions and show that these functions are nowhere Fréchet differentiable contrarily to what is asserted in some papers. Finally, Section 6 is devoted to the discussion of several results from the literature which are not convincing in our opinion.