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An analysis of the last round matching problem

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Abstract

The probability distribution of the number of players in the last round of a matching problem is analyzed and the existence of the limiting distribution is proved by using convolution method.

Keywords: Montmort random variable; Multi-round matching problem; Convolution

1. Introduction

The matching problem or the "Hats Problem" goes back to at least 1713 when it was proposed by the French mathematician Pierre de Montmort in his book [9] on games of gambling and chance, *Essay d'Analyse sur les Jeux de Hazard* (see also [11]). Although it has many formulations, the most common one is as follows:

Suppose that each of n men at a party throws his hat into the center of the room. The hats are first mixed up, and then each man randomly selects a hat. What is the probability that exactly k of the men select their own hats? And what is the expected number of people that select their own hats?

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There are several approaches such as the inclusion–exclusion principle, recurrence relations, binomial inversion, to find the distribution of the so called Montmort random variable X_n, the number of matches with n men. These approaches can be found, for example, in [10, pp. 125–127] and [4]. See more details in the next section. A celebrated paper of Kaplansky [7] first showed that X_n has an approximate Poisson distribution. Takács [12] gave an extensive history for this single round version of the matching problem. The problem of a multi-round matching is as follows:

Suppose that those choosing their own hats depart, while the others put their selected hats in the center of the room, mixed them up, and then reselect. Also, suppose that this process continues until each individual has his own hat. Assume we start with n men. What is the expected number τ_n of rounds that are necessary?

Detailed analysis can be found in [10, p. 110], see also Section 2 for more detail. The purpose of this note is to analyze the following questions in the multi-round matching problem:

What is the expected number of men, L_n, on the last round? And what is the limiting distribution of L_n as $n \to \infty$?

Note that the last round is the round that all remaining players obtained their own hats. This requires us to determine the distribution of the stopped outcome and hence it can be viewed as conditioning on the future. There are many problems of interests that involve the moments (outcomes) of the last events at a stopping time, and they are in general hard problems even in determination of the expected values (see [13, p. 162], for a case of independent random walks).

We will show in Section 3 that the limiting distribution of L_n exists and the expectation $\mathbb{E}(L_n)$ converges to a constant $l \approx 2.26264703816$. The exact value of the constant l is still unknown. Let p_n be the probability that any particular individual man (among n men) is in the last round. Then we have $p_n = \mathbb{E}(L_n)/n$ by taking the expectation on the relation $L_n = \sum_{i=1}^{n} 1_{A_i}$, where A_i is the event that the ith man is in the last round. Thus $p_n = \mathbb{E}(L_n)/n \sim l/n$ as $n \to \infty$ is the asymptotic probability of winning (a winner is defined to be one in the last round), and the fair pay off for a winner is $1/p_n \sim n/l \approx n/2.262647$ unit if each man pays a unit to play.

The remaining of the paper is organized as follows. In Section 2, we provide mathematical formulation of the problem together with some remarkable properties of the Montmort random variable X_n, and we give the recursive relations for the expectation $\mathbb{E}(L_n)$ and estimate the distribution $q_{n,m} = \mathbb{P}(L_n = m)$. We prove our main result in Section 3 by using convolution method. In Section 4, we provide a table for the distribution of L_n for $2 \leqslant n \leqslant 15$ and numerical approximations of the limiting values. Related problems, questions and remarks are collected at the end of Section 4.

2. Recursive relations

Mathematically, we are dealing with random permutation of n elements and the Montmort random variable X_n is the number of fixed elements in a random permutation. Its distribution is well known and is given by

$$p_{n,k} = \mathbb{P}(X_n = k) = \frac{1}{k!}\sum_{i=0}^{n-k}(-1)^i\frac{1}{i!}, \quad k = 0, 1, \ldots, n. \tag{2.1}$$

Feller [3, p. 231] showed that both the mean and the variance of X_n are equal to one for all $n \geqslant 2$. He also derived the Poisson approximation of rate $\lambda = 1$ for X_n. Some rather remarkable