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Asplund sets, differentiability and subdifferentiability of functions in Banach spaces ☆

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Abstract

We show that Asplund sets are effective tools to study differentiability of Lipschitz functions, and ε-subdifferentiability of lower semicontinuous functions on general Banach spaces. If a locally Lipschitz function defined on an Asplund generated space $X = \overline{TY}$ has a minimal Clarke subdifferential mapping, then it is $T\mathbb{B}_Y$-uniformly strictly differentiable on a dense G_δ subset of X. Examples are given of locally Lipschitz functions that are $T\mathbb{B}_Y$-uniformly strictly differentiable everywhere, but nowhere Fréchet differentiable.

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1. Introduction

This note is concerned with *Asplund sets* and its applications in differentiability of Lipschitz functions and subdifferentiability of lower semicontinuous functions on general Banach spaces. Roughly speaking, we show that if a set is Asplund in a Banach space, then Lipschitz functions are partially differentiable along the set densely; and for every $\varepsilon > 0$ lower semicontinuous functions are partially ε-subdifferentiable along the set densely in their domains. Our key tools are the characterization of Asplund sets by Fitzpatrick [4, p. 122], and the interpolation theorem of

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Asplund generated spaces by Stegall [9, pp. 22–26]. Sova's example [12] allows us to demonstrate that such a partial differentiability is still far away from the Fréchet differentiability in non-Asplund spaces. Our results are in the same spirit as the partial subdifferentiability results for lower semicontinuous functions given by Borwein, Treiman, and Zhu [1], in which they assume a Banach space X with a Banach subspace Y has a $Y\beta$-smooth norm. The Asplundity of a set in general Banach spaces turns out to be a surprisingly applicable concept in studying the existence of derivative and subderivatives of functions.

2. Basic properties of Asplund sets

Let X be a Banach space and (X^*, weak^*) be the dual of X with weak* topology. We use $\mathbb{B}_X$ and $\mathbb{B}_{X^*}$ to denote the closed unit balls in X and X^*, respectively. For a bounded absolutely convex set $M \subset X$, according to Fitzpatrick [4, p. 122], it is called *Asplund* if each bounded subset K of X^* is M-dentable. That is, for every $\varepsilon > 0$ there exists $0 \neq e \in X$ and $\nu > 0$ such that the slice

$$S(K, e, \nu) := \{x^* \in K\colon \langle x^*, e\rangle > \sup\langle K, e\rangle - \nu\},$$

has M-diameter

$$\operatorname{diam}_M S(K, e, \nu) := \sup\{\langle x^* - y^*, h\rangle\colon x^*, y^* \in S(K, e, \nu),\ h \in M\} < \varepsilon.$$

Finite sets and weakly compact sets (so compact sets) of any Banach spaces are Asplund sets. For a Banach space X, $\mathbb{B}_X$ is an Asplund set if and only if X is an Asplund space by Theorem 1.1.1 [9]. Every bounded subset of an Asplund space is an Asplund set. In particular, a set $M \subset X$ is Asplund if and only if for every separable subspace $Z \subset X$, whenever $Z \cap M \neq \emptyset$, the set $Z \cap M$ is Asplund in Z. This means that a set being Asplund is separably determined. Many other characterizations of Asplund sets may be found in [4,9,18].

Let $U \subset X$ be nonempty open. A function $f: U \to \mathbb{R}$ is called locally Lipschitz if given $x \in U$, $\exists L(x) > 0$, $\delta(x) > 0$ such that

$$|f(y) - f(z)| \leqslant L\|y - z\| \quad \text{for } y, z \in \mathbb{B}(x, \delta),$$

where $\mathbb{B}(x, \delta) := \{y \in X\colon \|y - x\| < \delta\}$. In order to study differentiability of locally Lipschitz functions, we need the *Clarke subdifferential* [5] defined by

$$\partial_c f(x) := \{x^*\colon \langle x^*, v\rangle \leqslant f^\circ(x; v) \text{ for all } v \in X\}, \tag{1}$$

where

$$f^\circ(x; v) := \limsup_{y \to x,\, t \downarrow 0} \frac{f(y + tv) - f(y)}{t}.$$

Such a function f is *M-differentiable* at $x \in U$ [4, p. 117] if there exists $x^* \in \partial_c f(x)$ satisfies

$$\lim_{t \to 0} \sup_{h \in M} \left| \frac{f(x + th) - f(x)}{t} - \langle x^*, h\rangle \right| = 0,$$

and we write $x^* \in f'_M(x)$. When $M := \mathbb{B}_X$, we say that f is *Fréchet differentiable*.

Asplund set is closely tied to the differentiability as the following illustrates:

Proposition 1. *Let X be a Banach space, $M \subset X$ be bounded. If M is non-Asplund, then there exists a norm $\|\cdot\|$ on X which is nowhere M-differentiable.*

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