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# Bounds of Hausdorff measure of the Sierpinski gasket ☆

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**Abstract**

By a new method, we obtain the lower and upper bounds of the Hausdorff measure of the Sierpinski gasket, which can approach the Hausdorff measure of the Sierpinski gasket infinitely.


*Keywords:* Hausdorff measure; Self-similar set; Sierpinski gasket

## 0. Introduction

The computation and estimation of the Hausdorff dimension and measure of the fractal sets are important problems in fractal geometry. Generally, the computation of the Hausdorff dimension, especially the Hausdorff measure, is very difficult. As a referee has previously pointed out, "Hausdorff measure is an important notion in the study of fractals. However there are few concrete results about computation of Hausdorff measure even for some simple fractals. Part of reason is the difficulty of the problem." For a self-similar set satisfying the open set condition, we know that its Hausdorff dimension equals its self-similar dimension [1]. However, there are not many results on the computation and estimation of Hausdorff measures for such fractal sets except for a few fractal sets on a line, like the Cantor set [2]. For the famous classical self-similar set, the Sierpinski gasket, its Hausdorff measure remains unknown. Nevertheless, efforts have been made in order to estimate the lower and upper bounds of its Hausdorff measure [3–8].

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In this paper, we develop a new method of estimating the upper bounds and lower bounds of the Hausdorff measure of the Sierpinski gasket. We show that the Hausdorff measure of the Sierpinski gasket can be squeezed out by sequences of lower bounds and upper bounds. Precisely speaking, we show that:

**Theorem.** *The Hausdorff measure of the Sierpinski gasket satisfies the estimation*

$$a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n} \leqslant H^s(S) \leqslant a_n, \quad \text{for } n \geqslant 1,$$

*where $a_n$ is defined in Proposition* 1.2.

The above theorem provides us a way, at least in theory, to estimate the Hausdorff measure of the Sierpinski gasket as close as we want.

In the end of the paper, we give two conjectures about $a_n$, for $n \geqslant 3$ and $H^s(S)$.

## 1. The Hausdorff measures of the self-similar sets

Let $D \subset R^n$ be a nonempty set. $E \subset R^n$ is a self-similar set defined by $m$ similar contracting maps $S_i : D \to D$, with contracting ratios, $0 < c_i < 1$ $(i = 1, 2, \ldots, m)$ and satisfying open set condition, that is, there exists a nonempty open set $U$ for which we have $S_i[U] \cap S_j[U] = \emptyset$ for $i \neq j$ and $U \supseteq S_i[U]$ for all $i$. Then $\dim_H(E) = s,\ 0 < H^s(E) < +\infty$.

Where $s$ satisfies $\sum_{i=1}^m c_i^s = 1$, $\dim_H(E)$ and $H^s(E)$ denote the Hausdorff dimension and measure of $E$, respectively. If $S_i[E] \cap S_j[E] = \emptyset,\ 0 < i < j \leqslant m$, we say that $E$ satisfies strong separate condition (SSC). Let $J_n = \{(i_1 i_2 \cdots i_n)\colon\ 1 \leqslant i_1, i_2, \ldots, i_n \leqslant m,\ n \geqslant 1\}$ and $E_{i_1 i_2 \cdots i_n} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(E)$ be self-similar $E$. It is easy to know that $E = \bigcup_{J_n} E_{i_1 i_2 \cdots i_n}$.

**Proposition 1.1.** [7] *Suppose that $E$ is a self-similar set satisfying the open set condition, then for any measurable set $U$, we have $H^s(E \cap U) \leqslant |U|^s$, where $s = \dim_H(E)$.*

**Proposition 1.2.** *Suppose that $E$ is a self-similar set satisfying the open set condition. For $n \geqslant 1$, $1 \leqslant k \leqslant m^n$, let $\Delta_1, \Delta_2, \ldots, \Delta_k \in \{E_{i_1 i_2 \cdots i_n}\colon\ 1 \leqslant i_1, i_2, \ldots, i_n \leqslant m\}$ and $\mu$ be the common self-similar probability measure on the $E$, $\mu(E_{i_1 i_2 \cdots i_n}) = c_{i_1}^s c_{i_2}^s \cdots c_{i_n}^s$.*

*Let*

$$b_k = \min_{\substack{\Delta_i \in \{E_{i_1 \cdots i_n}\} \\ i = 1, 2, \ldots, k}} \left\{ \frac{|\bigcup_{i=1}^k \Delta_i|^s}{\mu(\bigcup_{i=1}^k \Delta_i)} \right\},$$

*where the minimum is taken for all possible union of $k$ elements of $\{E_{i_1 i_2 \cdots i_n}\}$ and $a_n = \min_{1 \leqslant k \leqslant m^n}\{b_k\}$. If there exists a constant $A > 0$ such that $a_n \geqslant A(n = 1, 2, \ldots)$, then $H^s(E) \geqslant A$.*

**Proof.** By [1, p. 33], we can get the same values for Hausdorff measure and dimension if in the definition of $H_\delta^s(E)$ we use $\delta$-cover of just open set. So in the mass distribution principle of [1], we can replace any sets by any open sets. For any open set $V$, let $F_n = \bigcup_{E_{i_1 i_2 \cdots i_n} \subset V} E_{i_1 i_2 \cdots i_n}$. It is obvious that $F_n \subset F_{n+1}$, $\bigcup_{n=1}^{+\infty} F_n = E \cap V$. By the property of measure $\mu$ and the definitions of $a_n$, $b_k$, we get