Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Math. Anal. Appl. 328 (2007) 119–136
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Stability and regularity results for a size-structured population model

Jozsef Z. Farkas, Thomas Hagen *

Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA

Received 16 January 2006
Available online 12 June 2006
Submitted by M. Iannelli

Abstract

In the present paper a nonlinear size-structured population dynamical model with size and density dependent vital rate functions is considered. The linearization about stationary solutions is analyzed by semigroup and spectral methods. In particular, the spectrally determined growth property of the linearized semigroup is derived from its long-term regularity. These analytical results make it possible to derive linear stability and instability results under biologically meaningful conditions on the vital rates. The principal stability criteria are given in terms of a modified net reproduction rate.

Keywords: Population dynamics; Size-structured; Stability theory; Spectral analysis

1. Introduction

The dynamics of a size-structured population living in a closed territory can be described by the model equation

$$p_t(a,t) + \big(\gamma\big(a, P(t)\big)p(a,t)\big)_a = -\mu\big(a, P(t)\big)p(a,t), \quad 0 \leqslant a \leqslant m < \infty, \tag{1.1}$$

subject to the boundary condition

$$p(0,t) = \int_0^m \beta\big(a, P(t)\big)p(a,t)\,da, \quad t > 0, \tag{1.2}$$

* Corresponding author.
E-mail addresses: jzfarkas@gmail.com (J.Z. Farkas), thagen@memphis.edu (T. Hagen).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.05.032

and an initial condition of the form

$$p(a,0) = p_0(a). \tag{1.3}$$

Here, the function $p = p(a,t)$ denotes the density of individuals of size $a \in [0,m]$ at time $t \in [0,\infty)$ where $m > 0$ denotes the (finite) maximum size of any individual in the population. The model equation involves the vital rates $\mu = \mu(a,P)$—mortality, $\beta = \beta(a,P)$—fertility and $\gamma = \gamma(a,P)$—growth rate, which depend on the size a and on the total population quantity P, given by

$$P(t) = \int_0^m p(a,t)\,da \tag{1.4}$$

at time t. Consequently, Eq. (1.1) is nonlinear. Note that for simplicity we have normalized the size of any newborn individual to 0.

We make the following general assumptions on the vital rate functions:

$$\mu, \beta \in C^1\big([0,m] \times [0,\infty)\big), \quad \beta \geqslant 0, \tag{1.5}$$

$$\gamma \in C^2\big([0,m] \times [0,\infty)\big), \quad \gamma > 0. \tag{1.6}$$

These assumptions will suffice (and could actually be relaxed) to make the analysis of the linearized problem work. They are, however, generally not strong enough to prove global existence results for the nonlinear problem. In addition, for practical purposes several other biologically relevant assumptions (such as $\mu > 0$) will have to be imposed on these functions.

The population model treated here is equivalent to the one usually considered in the literature (see [1,2] and references therein) when the boundary condition (1.2) is replaced by

$$\gamma\big(0, P(t)\big)p(0,t) = \int_0^m \beta\big(a, P(t)\big)p(a,t)\,da, \quad t > 0, \tag{1.7}$$

and no population inflow from an external source takes place. Condition (1.2) incorporates the γ-term on the left of Eq. (1.7) in the birth rate β on the right. We prefer working with the boundary condition in the form of Eq. (1.2) to simplify the following developments. Local and global existence and uniqueness of solutions to this nonlinear problem have been analyzed in [2]. The model considered here reduces to the Gurtin–MacCamy (or McKendrick) nonlinear age-structured model if $\gamma \equiv 1$ (see [13]) and is a generalization of the simple problem treated in [10]. Similar physiologically structured population models have been studied intensively in the literature. Let us just mention the well-known works [20,22,25] for reference here.

The main purpose of the present work is to investigate the linear stability of stationary solutions of the system (1.1)–(1.3) using semigroup techniques and spectral methods based on the characteristic equation. Linear semigroup methods were successfully developed to study the linear stability and regularity of solutions of linearized fluid flow problems where the underlying dynamics is driven by a one-dimensional mass transport equation (see [14,18,19]). The model equations treated in this work are similar in nature. Sophisticated semigroup methods have recently been used to obtain sharp regularity results for one-dimensional hyperbolic–elliptic fluid flow problems (see [15,17]) and to explain the phenomenon that the roots of the underlying characteristic equations (eigenvalues) are lined up along certain curves (see [16]). Earlier, quite deep semigroup results for an n-dimensional age-structured model with constant growth rate can be