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On some applications of the Briot–Bouquet differential subordination

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Abstract

Recently Srivastava et al. [J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14 (2003) 7–18; J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999) 1–13; Y.C. Kim, H.M. Srivastava, Fractional integral and other linear operators associated with the Gaussian hypergeometric function, Complex Var. Theory Appl. 34 (1997) 293–312] introduced and studied a class of analytic functions associated with the generalized hypergeometric function. In the present paper, by using the Briot–Bouquet differential subordination, new results in this class are obtained.

Keywords: Analytic functions; The generalized hypergeometric function; The Carlson–Shaffer operator; The Briot–Bouquet differential subordination

1. Introduction

Let $\mathcal{A}$ denote the class of functions which are *analytic* in $\mathcal{U} = \mathcal{U}(1)$, where

$$\mathcal{U}(r) = \{z\colon\ z \in \mathbf{C} \text{ and } |z| < r\}.$$

We denote by $\mathcal{A}_0$ the class of functions $f \in \mathcal{A}$ with the normalization $f(0) = f'(0) - 1 = 0$.

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We say that a function $f \in \mathcal{A}$ is *subordinate* to a function $F \in \mathcal{A}$ and write $f(z) \prec F(z)$, if and only if there exists a function $\omega \in \mathcal{A}$,

$$\omega(0) = 0, \qquad \left|\omega(z)\right| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = F\big(\omega(z)\big) \quad (z \in \mathcal{U}).$$

Moreover, we say that f is subordinate to F in $\mathcal{U}(r)$, if $f(rz) \prec F(rz)$. We shall write

$$f(z) \prec_r F(z)$$

in this case. In particular, if F is univalent in $\mathcal{U}$, we have the following equivalence (cf. [10]):

$$f(z) \prec F(z) \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathcal{U}) \subset F(\mathcal{U}).$$

For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

by $f * g$ we denote the *Hadamard product or convolution* of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let $q, s \in \mathbf{N} = \{1, 2, \ldots\}$, $q \leqslant s + 1$. For complex parameters $a_1, \ldots, a_q$ and $b_1, \ldots, b_s$ ($b_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s$), we define the *generalized hypergeometric function* ${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ by

$${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \cdots (b_s)_n} \frac{z^n}{n!} \quad (z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbf{N}). \end{cases}$$

Corresponding to a function $h(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ defined by

$$h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = z\,{}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z),$$

we consider a linear operator

$$H(a_1, \ldots, a_q; b_1, \ldots, b_s) : \mathcal{A}_0 \to \mathcal{A}_0,$$

defined by the convolution:

$$H(a_1, \ldots, a_q; b_1, \ldots, b_s) f(z) = h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) * f(z).$$

In particular, for $s = 1$ and $q = 2$ and $a_2 = 1$, we have the Carlson–Shaffer operator

$$\mathcal{L}(a_1, b_1) f(z) = H_1(a_1, 1; b_1) f(z),$$

which was introduced by Carlson and Shaffer [1] (see also [8]).

After some calculations we obtain

$$aH(a + 1) f(z) = zH'(a) f(z) + (a - 1) H(a) f(z), \tag{1}$$