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Globally and locally attractive solutions for quasi-periodically forced systems

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Abstract

We consider a class of differential equations, $\ddot{x} + \gamma \dot{x} + g(x) = f(\boldsymbol{\omega} t)$, with $\boldsymbol{\omega} \in \mathbb{R}^d$, describing one-dimensional dissipative systems subject to a periodic or quasi-periodic (Diophantine) forcing. We study existence and properties of trajectories with the same quasi-periodicity as the forcing. For $g(x) = x^{2p+1}$, $p \in \mathbb{N}$, we show that, when the dissipation coefficient is large enough, there is only one such trajectory and that it describes a global attractor. In the case of more general nonlinearities, including $g(x) = x^2$ (describing the varactor equation), we find that there is at least one trajectory which describes a local attractor.

Keywords: Dissipative systems; Quasi-periodically forced systems; Varactor equation; Attractor; Global attractivity

1. Introduction

Consider the ordinary differential equation

$$\ddot{x} + \gamma \dot{x} + x^{2p+1} = f(\boldsymbol{\omega} t), \tag{1.1}$$

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where $p \in \mathbb{N}$, $\boldsymbol{\omega} \in \mathbb{R}^d$ is the frequency vector, $f(\boldsymbol{\psi})$ is an analytic quasi-periodic function,

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \tag{1.2}$$

with average $\langle f \rangle \equiv f_{\mathbf{0}} \neq 0$, and $\gamma > 0$ is a real parameter (dissipation coefficient). Here and henceforth we denote with $\cdot$ the scalar product in $\mathbb{R}^d$. By the analyticity assumption on f there are two strictly positive constants F and ξ such that one has $|f_{\boldsymbol{\nu}}| \leqslant F e^{-\xi|\boldsymbol{\nu}|}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$.

If $d > 1$ we shall assume a Diophantine condition on the frequency vector $\boldsymbol{\omega}$, that is

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geqslant C_0 |\boldsymbol{\nu}|^{-\tau} \quad \forall \boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \tag{1.3}$$

where $|\boldsymbol{\nu}| = |\boldsymbol{\nu}|_1 \equiv |\nu_1| + \cdots + |\nu_d|$, and C_0 and τ are positive constants, with $\tau > d - 1$ and C_0 small enough. Note that for $d = 1$ the condition (1.3) is automatically satisfied for all $\boldsymbol{\omega} \neq \mathbf{0}$.

In this paper we want to show that for γ large enough the system (1.1) admits a global attractor which is a quasi-periodic solution with the same frequency vector $\boldsymbol{\omega}$ as the forcing f. This will be done in two steps: first we prove that for γ large enough there is a quasi-periodic solution $x_0(t)$ with frequency vector $\boldsymbol{\omega}$ (cf. Theorem 1 in Section 2); second we prove that, again for γ large enough, any trajectory is attracted by $x_0(t)$ (cf. Theorem 2 in Section 3).

In particular, this solves for the system (1.1) a problem left as open in [12]. Indeed in [12] we considered a class of ordinary differential equations, including (1.1), and proved existence of a quasi-periodic solution with the same quasi-periodicity as the forcing, but we could not conclude, not even locally, that this was the only solution with such a property. The result stated above gives an affirmative answer to this problem for the system (1.1), by showing that the quasi-periodic solution $x_0(t)$ is unique; cf. Theorem 3 in Section 4.

This uniqueness result holds for the more general systems studied in [12], including the resistor–inductor–varactor circuit, or simply varactor equation, studied in [4,12]. This is a simple electronic circuit described by the equation $\ddot{x} + \gamma\dot{x} + x^\mu = f(\boldsymbol{\omega}t)$, for $x > 0$, where $R = \gamma$ is the resistance, $L = 1$ is the (normalised) inductance, $f(\boldsymbol{\omega}t)$ is the electromotive force, $v(t) = x(t)$ is the varactor voltage, and $i(t) = \dot{x}(t)$ is the current. The varactor is a particular type of diode, and it is described by the nonlinear term x^μ, where typically $\mu \in [1.5, 2.5]$. In [4,12] the case $\mu = 2$ was explicitly considered, for the sake of simplicity and concreteness. In these more general cases, the solution $x_0(t)$ is not a global attractor, but it turns out to be the only attractor in a neighbourhood of the solution itself.

More precisely the situation is as follows. We can consider systems described by

$$\ddot{x} + \gamma\dot{x} + g(x) = f(\boldsymbol{\omega}t), \tag{1.4}$$

where f is given by (1.2) and g is an analytic function. The case $g(x) = x^2$ corresponds to the varactor equation studied in [4]. Studying the behaviour of the system (1.4) for γ large enough suggests to introduce a new parameter $\varepsilon = 1/\gamma$, in terms of which the differential equation (1.4) becomes

$$\varepsilon\ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\boldsymbol{\omega}t), \tag{1.5}$$

and study what happens for ε small enough.

If we assume that there exists $c_0 \in \mathbb{R}$ such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) := \partial_x g(c_0) \neq 0$, then the system (1.5) admits a quasi-periodic solution $x_0(t)$, analytic in t, with the same frequency vector $\boldsymbol{\omega}$ as the forcing f, and furthermore $x_0(t) = c_0 + O(\varepsilon)$. This was proved in [12], where