Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 331 (2007) 686–700
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Inclusion relations and convolution properties of a certain class of analytic functions associated with the Ruscheweyh derivatives ☆

H.M. Srivastava [a,*], N-Eng Xu [b], Ding-Gong Yang [c]

[a] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*
[b] *Department of Mathematics, Changshu Institute of Technology, Changshu, Jiangsu 215500, People's Republic of China*
[c] *Department of Mathematics, Suzhou University, Suzhou, Jiangsu 215006, People's Republic of China*

Received 9 May 2006

Available online 6 October 2006

Submitted by S. Ruscheweyh

Abstract

Let $\mathcal{A}$ denote the class of functions $f(z)$ with

$$f(0) = f'(0) - 1 = 0,$$

which are analytic in the open unit disk $\mathbb{U}$. By means of the Ruscheweyh derivatives, we introduce and investigate the various properties and characteristics of a certain two-parameter subclass $\mathcal{T}(\alpha, \lambda; h)$ of $\mathcal{A}$, where $\alpha \geqq 0$, $\lambda > -1$, and $h(z)$ is analytic and convex univalent in $\mathbb{U}$ with $h(0) = 1$. In particular, some inclusion relations and convolution properties for the function class $\mathcal{T}(\alpha, \lambda; h)$ are presented here.

Keywords: Analytic functions; Convex functions; Univalent functions; Close-to-convex functions; Hadamard product (or convolution); Subordination between analytic functions; Ruscheweyh derivatives

☆ The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

* Corresponding author.
E-mail addresses: harimsri@math.uvic.ca (H.M. Srivastava), xuneng11@pub.sz.jsinfo.net (N-E. Xu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.09.019

1. Introduction, definitions and preliminaries

Let the functions

$$f(z)=\sum_{k=0}^{\infty}a_k z^k \quad \text{and} \quad g(z)=\sum_{k=0}^{\infty}b_k z^k \tag{1.1}$$

be *analytic* in the *open* unit disk

$$\mathbb{U}=\{z\colon\ z\in\mathbb{C} \text{ and } |z|<1\}.$$

Then the Hadamard product (or convolution) $(f*g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z):=\sum_{k=0}^{\infty}a_k b_k z^k=:(g*f)(z).$$

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1.2}$$

which are *analytic* in $\mathbb{U}$. A function $f(z)\in\mathcal{A}$ is said to be *starlike of order* β in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\beta \quad (z\in\mathbb{U}) \tag{1.3}$$

for some β $(\beta<1)$. We denote this class by $\mathcal{S}^*(\beta)$. A function $f(z)\in\mathcal{A}$ is said to be *prestarlike of order* β $(\beta<1)$ in $\mathbb{U}$ if

$$\frac{z}{(1-z)^{2(1-\beta)}}*f(z)\in\mathcal{S}^*(\beta). \tag{1.4}$$

We denote this class by $\mathcal{R}(\beta)$.

Next we define the Ruscheweyh derivative operator D^λ by

$$D^\lambda f(z):=\frac{z}{(1-z)^{\lambda+1}}*f(z) \quad (f\in\mathcal{A};\ \lambda>-1). \tag{1.5}$$

In particular, for

$$\lambda=n \quad \bigl(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\ \mathbb{N}:=\{1,2,3,\ldots\}\bigr),$$

we easily find from the definition (1.5) that

$$D^n f(z)=\frac{z\bigl(z^{n-1}f(z)\bigr)^{(n)}}{n!} \quad (n\in\mathbb{N}_0). \tag{1.6}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $f(z)$ is *subordinate* to $g(z)$ in $\mathbb{U}$, and we write $f(z)\prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that

$$\bigl|w(z)\bigr|\leqq|z| \quad \text{and} \quad f(z)=g\bigl(w(z)\bigr) \quad (z\in\mathbb{U}).$$

If $g(z)$ is univalent in $\mathbb{U}$, then the following equivalence relationship holds true:

$$f(z)\prec g(z) \quad \Longleftrightarrow \quad f(0)=g(0) \quad \text{and} \quad f(\mathbb{U})\subset g(\mathbb{U}).$$