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Briot–Bouquet differential superordinations and sandwich theorems

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Abstract

Briot–Bouquet differential subordinations play a prominent role in the theory of differential subordinations. In this article we consider the dual problem of Briot–Bouquet differential superordinations. Let β and γ be complex numbers, and let Ω be any set in the complex plane $\mathbf{C}$. The function p analytic in the unit disk $\mathbf{U}$ is said to be a *solution* of the *Briot–Bouquet differential superordination* if

$$\Omega \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in U \right\}.$$

The authors determine properties of functions p satisfying this differential superordination and also some generalized versions of it.

In addition, for sets Ω_1 and Ω_2 in the complex plane the authors determine properties of functions p satisfying a Briot–Bouquet sandwich of the form

$$\Omega_1 \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in U \right\} \subset \Omega_2.$$

Generalizations of this result are also considered.
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1. Introduction

We begin by introducing the two important classes of functions considered in this article.

Let $\mathbf{H} = \mathbf{H}(\mathbf{U})$ denote the class of functions analytic in $\mathbf{U}$. For n a positive integer and $a \in \mathbf{C}$, let

$$\mathbf{H}[a,n] = \{ f \in \mathbf{H} \mid f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.$$

Let $\mathbf{Q}$ denote the set of functions f that are analytic and injective on the set $\overline{\mathbf{U}} \setminus \mathbf{E}(f)$, where

$$\mathbf{E}(f) = \Big\{ \varsigma \in \partial U \;\Big|\; \lim_{z \to \varsigma} f(z) = \infty \Big\},$$

and are such that $f'(\varsigma) \neq 0$ for $\varsigma \in \partial \mathbf{U} \setminus \mathbf{E}(f)$. The subclass of $\mathbf{Q}$ for which $f(0) = a$ is denoted by $\mathbf{Q}(a)$.

Most of the functions considered in this article, and conditions on them are defined uniformly in the unit disk $\mathbf{U}$. Because of this we shall omit the requirement "$z \in \mathbf{U}$" in most of the definitions and results.

Many of the inclusion results that follow can be written very neatly in terms of subordination and superordination. We recall these definitions. Let $f, F \in \mathbf{H}$ and let F be univalent in $\mathbf{U}$. The function F is said to be *superordinate to* f, or f is *subordinate to* F, written $f \prec F$, if $f(0) = F(0)$ and $f(\mathbf{U}) \subset F(\mathbf{U})$.

Let β and γ be complex numbers, let Ω_2 and Δ_2 be sets in the complex plane, and let p be analytic in the unit disk $\mathbf{U}$. In a series of articles the authors and many others [7, pp. 80–119] have determined conditions so

$$\left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in \mathbf{U} \right\} \subset \Omega_2 \quad \Rightarrow \quad p(\mathbf{U}) \subset \Delta_2. \tag{1}$$

The differential operator on the left is known as the *Briot–Bouquet differential operator*. The main concern in this subject has been to find the smallest set Δ_2 in $\mathbf{C}$ for which (1) holds. This particular differential implication has a surprising number of applications in univalent function theory.

In this article we consider the dual problem of determining conditions so that

$$\Omega_1 \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\middle|\; z \in \mathbf{U} \right\} \quad \Rightarrow \quad \Delta_1 \subset p(\mathbf{U}). \tag{2}$$

In particular, we are interested in determining the largest set Δ_1 in $\mathbf{C}$ for which (2) holds.

If the sets Ω and Δ in (1) and (2) are simply connected domains not equal to $\mathbf{C}$, then it is possible to rephrase these expressions very neatly in terms of subordination and superordination in the forms:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \quad \Rightarrow \quad p(z) \prec q_2(z), \tag{1'}$$

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \quad \Rightarrow \quad q_1(z) \prec p(z). \tag{2'}$$

The left side of (1′) is called a *Briot–Bouquet differential subordination*, and the function q_2 is called a *dominant* of the differential subordination. The *best dominant,* which provides a sharp result, is the dominant that is subordinate to all other dominants. Many results and applications on these topics can be found in [7, pp. 80–119].

In a recent paper [6] the authors have introduced the dual concept of a differential superordination. In light of those results we call the left side of (2′) a *Briot–Bouquet differential*

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