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A functional equation originating from quadratic forms ☆

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Abstract

In this paper, we obtain the general solution and the stability of the 2-variable quadratic functional equation

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w).$$

The quadratic form $f(x,y)=ax^2+bxy+cy^2$ is a solution of the above functional equation.

Keywords: Solution; Stability; 2-Variable quadratic functional equation

1. Introduction

A mapping f is called a *quadratic form* if there exist $a,b,c\in\mathbb{R}$ such that

$$f(x,y)=ax^2+bxy+cy^2$$

for all $x,y\in X$.

In this paper, let X and Y be real vector spaces. For a mapping $f:X\times X\to Y$, consider the 2-variable quadratic functional equation:

$$f(x+y,z+w)+f(x-y,z-w)=2f(x,z)+2f(y,w). \tag{1}$$

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When $X = Y = \mathbb{R}$, the quadratic form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := ax^2 + bxy + cy^2$ is a solution of (1).

For a mapping $g : X \to Y$, consider the quadratic functional equation:

$$g(x + y) + g(x - y) = 2g(x) + 2g(y). \tag{2}$$

In 1989, J. Aczél [1] solved the solution of Eq. (2). Later, many different quadratic functional equations were solved by numerous authors [2–5].

In this paper, we investigate the relation between (1) and (2). And we find out the general solution and the generalized Hyers–Ulam stability of (1).

2. Results

The 2-variable quadratic functional equation (1) induces the quadratic functional equation (2) as follows.

Theorem 1. *Let $f : X \times X \to Y$ be a mapping satisfying* (1) *and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x, x) \tag{3}$$

for all $x \in X$, then g satisfies (2).

Proof. By (1) and (3),

$$\begin{aligned} g(x + y) + g(x - y) &= f(x + y, x + y) + f(x - y, x - y) \\ &= 2f(x, x) + 2f(y, y) \\ &= 2g(x) + 2g(y) \end{aligned}$$

for all $x, y \in X$. $\square$

Example 1. Let X be a real algebra and $D : X \to X$ a derivation on X. Define a mapping $f : X \times X \to X$ by

$$f(x, y) := D(xy) = xD(y) + D(x)y$$

for all $x, y \in X$. Then f satisfies (1). Define a mapping $g : X \to X$ by

$$g(x) := D(x^2) = xD(x) + D(x)x$$

for all $x \in X$. Then g satisfies (3). By Theorem 1, g satisfies (2).

The quadratic functional equation (2) induces the 2-variable quadratic functional equation (1) with an additional condition.

Theorem 2. *Let $a, b, c \in \mathbb{R}$ and $g : X \to Y$ be a mapping satisfying* (2). *If $f : X \times X \to Y$ is the mapping given by*

$$f(x, y) := ag(x) + \frac{b}{4}\big[g(x + y) - g(x - y)\big] + cg(y) \tag{4}$$

for all $x, y \in X$, then f satisfies (1). *Furthermore,* (3) *holds if $a + b + c = 1$.*