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Symmetrically localized frames and the removal of subsets of positive density

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Abstract

The theorems of Balan, Casazza, Heil, and Landau concerning the removal of sets of positive density from frames with positive excess are extended using a more general, symmetric concept of localization of frames.

Keywords: Frames; Localized frames; Density; Excess; Overcompleteness; Modulation spaces; Gabor systems; Weyl–Heisenberg systems

1. Introduction

The concept of localization of frames was recently introduced independently by Gröchenig [18] and the group consisting of Balan, Casazza, Heil, and Landau (BCHL) [4,5]. Frames having this new localization property are interesting in a number of ways; Gröchenig proved that a frame localized with respect to a Riesz basis is automatically a Banach frame for an often important family of Banach spaces associated to the Riesz basis. This was further generalized in [12–14, 17]. Further background and examples can be found in [1,7,11]. BCHL proved that the excess of an overcomplete localized frame has a certain degree of uniformity, and were able to give conditions under which excess of positive density could be removed from an overcomplete localized frame. Previous work done in this direction can be found in [2,3]. As often happens when a concept is introduced independently by several parties, the definitions found in [4,5,18] are

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different enough so that one definition is not a special case of the other, and are specific to their respective purposes. A more general definition which encompasses both Gröchenig's and BCH-L's definitions in the most useful cases is introduced in [15]. This definition allows for a useful and natural equivalence class structure when dealing with l^1-self-localized frames, and extends the results of Gröchenig [15]. In this paper, we focus on extending the results of BCHL involving removing excess of positive density. In Section 2, we define symmetric localization. In Section 3, we provide the necessary definitions from [2]. In Section 4 we extend the Density-Relative Measure Theorem and the theorem concerning the removal of sets of positive density.

2. Symmetric localization

Before introducing the new definition, we fix basic notation. We recommend [6,8–10,16] for additional background.

Let $\mathcal{F} = \{f_x\}_{x\in X}$ be a *frame* for a separable Hilbert space $\mathcal{H}$ with frame bounds A, B. The *analysis operator* will be denoted $C : \mathcal{H} \to l^2(X)$, $C(f) = \{\langle f, f_x\rangle\}_{x\in X}$. The *synthesis operator* denoted $D : l^2(X) \to \mathcal{H}$, $D(\{c_x\}_{x\in X}) = \sum c_x f_x$ is the adjoint of C, $D = C^*$. The *frame operator* denoted $S = DC : \mathcal{H} \to \mathcal{H}$, $Sf = \sum_{x\in X}\langle f, f_x\rangle f_x$ is a positive, invertible operator such that $A\cdot I \leqslant S \leqslant B\cdot I$. The *canonical dual frame* of $\mathcal{F}$ will be denoted $\tilde{\mathcal{F}} = \{\tilde{f}_x\}_{x\in X} := \{S^{-1}f_x\}_{x\in X}$. A *frame sequence* $\mathcal{F} = \{f_x\}_{x\in X}$ is a frame for the closure of its span.

In the following, let G be a group of the form $\prod_{i=1}^d a_i\mathbb{Z} \times \prod_{j=1}^e \mathbb{Z}_{b_j}$. For every $g = (a_1n_1, a_2n_2, \ldots, a_dn_d, m_1, m_2, \ldots, m_e) \in G$, let

$$|g| = \sup\bigl\{|a_1n_1|, |a_2n_2|, \ldots, |a_dn_d|, \delta(m_1), \delta(m_2), \ldots, \delta(m_e)\bigr\}$$

where

$$\delta(m_j) = \begin{cases} 0 & \text{if } m_j = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Define a metric on G by $d(g, h) = |g - h|$ for $g, h \in G$. Let $S_n(j)$ denote the ball of radius n centered at j in G and $|S_n(j)| := \#[S_n(j)]$, the cardinality of $S_n(j)$.

Definition 1 *(Symmetric localization).* Let $\mathcal{F} = \{f_x\}_{x\in X}$ and $\mathcal{E} = \{e_y\}_{y\in Y}$ be sequences in a Hilbert space $\mathcal{H}$, X and Y arbitrary index sets.

(1) $(\mathcal{F}, \mathcal{E})$ is *symmetrically l^p-localized* if there exist maps $a_X : X \to G$, $a_Y : Y \to G$ such that $\sup_{j\in G} |a_X^{-1}(j)|, \sup_{j\in G} |a_Y^{-1}(j)| \leqslant K < \infty$, and $r \in l^p(G)$ such that for all $x \in X, y \in Y$,

$$\bigl|\langle f_x, e_y\rangle\bigr| \leqslant r_{a_X(x)-a_Y(y)}.$$

(2) $(\mathcal{F}, \mathcal{E})$ has *uniform l^p-column decay* if for every $\epsilon > 0$ there is $N_\epsilon > 0$ such that for all $y \in Y$,

$$\sum_{x\in X\setminus a_X^{-1}(S_{N_\epsilon}(a_Y(y)))} \bigl|\langle f_x, e_y\rangle\bigr|^p < \epsilon.$$

(3) $(\mathcal{F}, \mathcal{E})$ has *uniform l^p-row decay* if for every $\epsilon > 0$ there is $N_\epsilon > 0$ such that for all $x \in X$,

$$\sum_{y\in Y\setminus a_Y^{-1}(S_{N_\epsilon}(a_X(x)))} \bigl|\langle f_x, e_y\rangle\bigr|^p < \epsilon.$$

Remark 2. The terms column and row decay come from considering the cross-Grammian matrix $[\langle f_x, e_y\rangle]_{X,Y}$.