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J. Math. Anal. Appl. 320 (2006) 902–915

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation

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Received 17 December 2004
Available online 1 September 2005
Submitted by J. Lavery

Abstract

In this paper, we consider the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}$$

with initial conditions and Dirichlet boundary conditions. For nonincreasing positive functions g and for $p > m$, we prove that there are solutions with positive initial energy that blow up in finite time.

Keywords: Blow-up; Finite time; Hyperbolic; Nonlinear damping; Positive initial energy; Viscoelastic

1. Introduction

In this paper, we are concerned with the initial-boundary-value problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t|u_t|^{m-2} = u|u|^{p-2}, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.022

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geqslant 1$) with a smooth boundary $\partial\Omega$, $p > 2$, $m \geqslant 1$, and g is a positive function. In the absence of the viscoelastic term (that is, if $g = 0$), the equation in (1.1) reduces to the nonlinearly damped wave equation

$$u_{tt} - \Delta u + u_t|u_t|^{m-2} = u|u|^{p-2}.$$

This equation has been extensively studied by many mathematicians. It is well known that in the further absence of the damping mechanism $u_t|u_t|^{m-2}$, the source term $u|u|^{p-2}$ causes finite-time blow-up of solutions with negative initial energy (see [1,9]). In contrast, in the absence of the source term, the damping term assures global existence for arbitrary initial data (see [8,10]). The interaction between the damping and source terms was first considered by Levine [11,12] for linear damping ($m = 2$). Levine showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to nonlinear damping ($m > 2$). In their work, the authors introduced a new method and determined relations between m and p for which there is global existence and other relations between m and p for which there is finite-time blow-up. Specifically, they showed that solutions with negative energy continue to exist globally if $m \geqslant p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. Messaoudi [15] extended the blow-up result of [7] to solutions with only negative initial energy. For related results, we refer the reader to Levine and Serrin [13], Levine and Ro Park [14], Vitillaro [19], Yang [20] and Messaoudi and Said-Houari [18].

In the presence of the viscoelastic term ($g \neq 0$), Cavalcanti et al. [4] studied (1.1) for $m = 2$ and a localized damping mechanism $a(x)u_t$ ($a(x)$ null on a part of the domain). They obtained an exponential rate of decay by assuming that the kernel g is of exponential decay. This work was later improved by Cavalcanti et al. [6] and Berrimi and Messaoudi [2] using different methods. In related work, Cavalcanti et al. [3] studied solutions of

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \gamma\Delta u_t = 0, \quad x \in \Omega,\ t > 0,$$

for $\rho > 0$ and proved a global existence result for $\gamma \geqslant 0$ and an exponential decay result for $\gamma > 0$. This latter result was extended by Messaoudi and Tatar [16] to a situation where a source term is competing with the damping induced by $-\gamma\Delta u_t$ and the integral term. Also, Cavalcanti et al. [5] established an existence result and a decay result for viscoelastic problems with nonlinear boundary damping.

Concerning nonexistence, Messaoudi [17] showed that Todorova and Georgiev's results can be extended to (1.1) using the technique of [7] with a modification in the energy functional due to the different nature of the problems.

In this article, we improve our earlier result by adopting and modifying the method of [19]. In particular, we will show that there are solutions of (1.1) with positive initial energy that blow up in finite time.

We first state a local existence theorem that can be established by combining arguments of [4,7].