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Geometry of cones and an application in the theory of Pareto efficient points ☆

Christos Kountzakis, Ioannis A. Polyrakis *

Department of Mathematics, National Technical University of Athens, Zografou 157 80, Athens, Greece

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Abstract

In this article we give a new criterion for the existence of a bounded base for a cone P of a normed space X. Also, if P is closed, we give a partial answer to the problem: is 0 a point of continuity of P if and only if 0 is a denting point of P? The above problems have applications in the theory of Pareto efficient points.

Keywords: Bases for cones; Denting points; Pareto efficient points; Schur property; Points of continuity; Quasi-interior points

1. Introduction

In this article we study the geometry of cones of normed spaces and we apply these results in the theory of vector optimization. Specifically we suppose that P is a cone of a normed space X and we study the existence of a bounded base for P, the 0-Schur property and the geometry of 0 (point of continuity, denting point) whenever it is an extreme point of P. It is known that the existence of a bounded base for P which is defined (the base) by

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* Corresponding author.
E-mail address: ypoly@math.ntua.gr (I.A. Polyrakis).

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a continuous linear functional of X is equivalent to the existence of interior points in the dual cone P^0 of P. In this article we prove that the existence of such a base for the cone P is equivalent to cone P having both the 0-Schur property (P has the 0-Schur property if each weakly convergent to zero sequence of P is also norm convergent) and the existence of quasi-interior positive elements in X^*. Therefore the existence of a bounded base for P is decomposed in two important properties of ordered spaces: the 0-Schur property and the existence of quasi-interior positive elements in the dual space of X.

The existence of a closed, bounded base for a cone is important in the theory of vector optimization. In [6] the notion of the super-efficient point of a subset is introduced by Borwein and Zhuang and many important results are proved in the case where the ordering cone has a bounded base. Also in [10] the notion of the nuclear cone (this type of cone is a generalization of a cone with a bounded base) in locally convex spaces is introduced by G. Isac and problems of vector optimization in connection with these cones are studied. Therefore our new criterion for the existence of a bounded base for a cone has applications in the theory of Pareto optimization.

Another property of cones which is important in this theory is the geometry of 0 (point of continuity, denting point) whenever it is an extreme point of P. Specifically in [7, Theorem 3.1], it is proved by X.H. Gong that if x_0 is an efficient point of a weakly compact, convex subset A of X and at least one of the following conditions is satisfied:

(i) $0 \notin \overline{P \setminus B(0,\rho)}^w$, for any $\rho \in (0,1)$, where $B(0,\rho)$ is the open ball of X of center 0 and radius ρ,
(ii) 0 is a denting point of P,
(iii) x_0 is a denting point of A,

then x_0 belongs to the closure of the set of positive proper efficient points of A.

Recall that (ii) is equivalent to the existence of a closed, bounded base for the cone P defined (the base) by a continuous linear functional and that (i) holds if and only if 0 is a point of continuity of P. In p. 624 of [7], X.H. Gong remarks that in reflexive spaces (i) and (ii) are equivalent and he questions if these conditions are equivalent in normed spaces.

In [12] it is proved by P.K. Lin, B.L. Lin and S.L. Troyanski that a point x_0 of a convex, closed bounded subset K of a Banach space X is a denting point of K if and only if x_0 is a point of continuity of K and also x_0 is an extreme point of K. Therefore by this result it follows easily that in Banach spaces (i) and (ii) are equivalent, i.e., 0 is a denting point of a closed cone P of a Banach space if and only if 0 is a point of continuity of P. In this article we give a partial answer to the above problem. Specifically we show, Theorem 4, that if P is a cone of a normed space X and the dual cone P^0 of P has quasi-interior points then 0 is a point of continuity of P if and only if 0 is a denting point of P. Note that in the above result we prove the equivalence without the assumption that the cone P is closed. Therefore if X is a Banach space and P^0 has quasi-interior points our result generates the existing result in [12] (for cones) because it shows the equivalence without the assumption that the cone P is closed. Finally note that cones with the 0-Schur property have been studied in [2,9].

Let X be a normed space. Denote by X^* the norm dual of X and by $\mathbb{R}_+$ the set of positive real numbers $\lambda \geqslant 0$. A nonempty, convex subset P of X is a **cone** if $\lambda P \subseteq P$ for