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J. Math. Anal. Appl. 320 (2006) 381–395
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Characterization of the closedness of the sum of two shift-invariant spaces ☆

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Received 11 April 2005
Available online 19 August 2005
Submitted by P.G. Casazza

Abstract

We first present a formula for the supremum cosine angle between two closed subspaces of a separable Hilbert space under the assumption that the 'generators' form frames for the subspaces. We then characterize the conditions that the sum of two, not necessarily finitely generated, shift-invariant subspaces of $L^2(\mathbb{R}^d)$ be closed. If the fibers of the generating sets of the shift-invariant subspaces form frames for the fiber spaces a.e., which is satisfied if the shift-invariant subspaces are finitely generated or if the shifts of the generating sets form frames for the respective subspaces, then the characterization is given in terms of the norms of possibly infinite matrices. In particular, if the shift-invariant subspaces are finitely generated, then the characterization is given wholly in terms of the norms of finite matrices.

Keywords: Shift-invariant space; Angle between subspaces; Multiresolution analysis; Pseudo-inverse

☆ This work was supported by Korea Research Foundation Grant KRF-2002-070-C00004.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.06.097

1. Introduction

For closed subspaces U and V of a Hilbert space $\mathcal{H}$ the *supremum cosine angle between U and V* is defined by

$$S(U,V) := \sup_{u\in U\setminus\{0\}} \left\{ \frac{\|P_V u\|}{\|u\|} \right\} = \|P_V|_U\|, \tag{1.1}$$

where P_V denotes the orthogonal projection of $\mathcal{H}$ onto V and $P_V|_U$ its restriction on U. It is shown in [30] that

$$S(U,V) = S(V,U) = \sup\left\{ \frac{|\langle u,v\rangle|}{\|u\|\|v\|}\colon\ u\in U\setminus\{0\},\ v\in V\setminus\{0\} \right\}.$$

The arc-cosine value of $S(U,V)$ is interpreted as the 'smallest angle' between the vectors from U and V [30]. We use the convention that $S(U,V)=0$ if either U or V is trivial. If we take the infimum instead of the supremum in the right-hand side of (1.1), then we have the *infimum cosine angle* $R(U,V)$. The two angles are related by the formula: $R(U,V) = (1-S(U,V^{\perp})^2)^{1/2}$ [30]. See [10] for an application of these angles to the perturbation of frame sequences.

The infimum cosine angle is closely related with the bi-orthogonality of two multi-resolution analyses (MRAs) [1,2,7,21–23,29–31]. In particular, the authors have recently found a useful expression of the infimum cosine angle between finitely generated shift-invariant subspaces in terms of the Gramians of generating sets in a companion paper [23]. On the other hand, the supremum cosine angle is closely related with the closedness of the sum of two closed subspaces of a Hilbert space, as can be seen in the following proposition by Tang.

Proposition 1.1. [29] *Let U and V be closed subspaces of a separable Hilbert space. Then $U+V$ is closed and $U\cap V=\{0\}$ if and only if $S(U,V)<1$.*

The previous proposition is [29, Theorem 2.1]. It is stated there that the proposition is probably a folk result. In [20], motivated by the problem of the association between wavelets and MRAs, Kim et al. addressed the problem of determining when the sum of two principal, i.e., singly generated, shift-invariant subspaces of $L^2(\mathbb{R})$ is closed [20, Theorem 2.3] (cf. Corollary 3.10). The proof therein, however, is rather complicated and is not easily generalized to multiply generated shift-invariant subspaces. In this article, we present a complete characterization of the closedness of the sum of two, not necessarily finitely generated, shift-invariant subspaces of $L^2(\mathbb{R}^d)$ under mild condition on the generating sets of the shift-invariant spaces by using Proposition 1.1 and the newly found method in [23].

In order to give an application of our results we present the following results of the authors in [24] without proof. We refer to Section 2 for the terminologies and basic facts of the theory of shift-invariant subspaces, and refer to [9,13,15] for the basic facts about the multiresolution analysis. The following is the canonical example of the 'wavelet tight frame' constructed from the 'unitary extension principle' of Ron and Shen [26]. See Sec-