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A general iterative method for nonexpansive mappings in Hilbert spaces

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Abstract

Let H be a real Hilbert space. Consider on H a nonexpansive mapping T with a fixed point, a contraction f with coefficient $0 < \alpha < 1$, and a strongly positive linear bounded operator A with coefficient $\bar{\gamma} > 0$. Let $0 < \gamma < \bar{\gamma}/\alpha$. It is proved that the sequence $\{x_n\}$ generated by the iterative method $x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n)$ converges strongly to a fixed point $\tilde{x} \in \mathrm{Fix}(T)$ which solves the variational inequality $\langle (\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leqslant 0$ for $x \in \mathrm{Fix}(T)$.

Keywords: Nonexpansive mapping; Iterative method; Variational inequality; Fixed point; Projection; Viscosity approximation

1. Introduction

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [1,4,5,7,8] and the references therein. A typical

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problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. Assume A is strongly positive; that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geqslant \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{2}$$

Recall that $T : H \to H$ is nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is the set $\mathrm{Fix}(T) := \{x \in H\colon\ Tx = x\}$. We assume that $\mathrm{Fix}(T) \neq \emptyset$ and $C = \mathrm{Fix}(T)$. It is well known that $\mathrm{Fix}(T)$ is closed convex (cf. [2]). In [5] (see also [7]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \quad n \geqslant 0, \tag{3}$$

converges strongly to the unique solution of the minimization problem (1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions that will be made precise in Section 3.

On the other hand, Moudafi [3] introduced the viscosity approximation method for nonexpansive mappings (see [6] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n), \quad n \geqslant 0, \tag{4}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved [3,6] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (4) strongly converges to the unique solution x^* in C of the variational inequality

$$\big\langle (I - f)x^*, x - x^*\big\rangle \geqslant 0, \quad x \in C. \tag{5}$$

In this paper we will combine the iterative method (3) with the viscosity approximation method (4) and consider the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geqslant 0. \tag{6}$$

We will prove in Section 3 that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution of the variational inequality

$$\big\langle (A - \gamma f)x^*, x - x^*\big\rangle \geqslant 0, \quad x \in C, \tag{7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).