Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 318 (2006) 112–119
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Approximation of anisotropic classes by wavelets

Jiang Yanjie

Department of Applied Mathematics, North China Electric Power University, Baoding 071003, People's Republic of China

Received 28 February 2005

Available online 13 June 2005

Submitted by M. Milman

Abstract

In this paper, we study the best approximation for anisotropic Sobolev and Besov classes in the $L_q(R^d)$ metric by wavelets and obtain some asymptotic estimates of approximation order.

Keywords: Anisotropic classes; Decomposition of functions; Best approximation; Regular wavelets

1. Introduction

Let R, Z, Z_+, and N be the sets of all real numbers, integers, non-negative integers, and positive integers, respectively. For $r = (r_1, r_2, \dots, r_d) \in Z_+^d$, let $D^{r_i} f(t) := \frac{\partial^{r_i} f(t)}{\partial t_i^{r_i}}$, $i = 1, \dots, d$, be the generalized derivative of f in the sense of Liouville. Then the anisotropic Sobolev space $L_p^r(R^d)$ $(1 \leqslant p \leqslant \infty)$ is defined as follows:

$$L_p^r(R^d) := \left\{ f \in L_p(R^d) \;\middle|\; \|f\|_{L_p^r} = \|f\|_p + \sum_{j=1}^{d} \left\| D^{r_i} f \right\|_p < \infty \right\}.$$

E-mail address: jiangyj@heinfo.net.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.05.034

For $k_i \in N$, $t_i \in R$, denote by

$$\Delta_{t_i}^{k_i} f(x) := \sum_{l=0}^{k_i} (-1)^{l+k_i} \binom{k_i}{l} f(x_1, \ldots, x_i + lt_i, \ldots, x_d)$$

the k_ith difference of f at the point x for x_i with step t_i, $i = 1, \ldots, d$.

Definition. Let $k = (k_1, \ldots, k_d) \in Z_+^d$, $r = (r_1, \ldots, r_d)$, $k_i > r_i > 0$, $1 \leqslant \theta \leqslant \infty$, and $1 \leqslant p < \infty$. We say that $f \in B_{p\theta}^r(R^d)$ if the function f satisfies the conditions:

(i) $f \in L_p(R^d)$,

(ii) $$\|f\|_{b_{x_j p\theta}^{r_j}(R^d)} := \begin{cases} \left\{\int_R \left(\frac{\|\Delta_{t_j}^{k_j} f(\cdot)\|_p}{|t_j|^{r_j}}\right)^\theta \frac{dt_j}{|t_j|}\right\}^{1/\theta} < \infty, & 1 \leqslant \theta < \infty, \\ \sup_{t_j \neq 0} \frac{\|\Delta_{t_j}^{k_j} f(\cdot)\|_{\mathbf{p}}}{|t_j|^{r_j}} < \infty, & \theta = \infty, \end{cases}$$

for $j = 1, \ldots, d$.

By [6], the linear space $B_{p\theta}^r(R^d)$ is a Banach space with the norm

$$\|f\|_{B_{p\theta}^r(R^d)} := \|f\|_p + \sum_{j=1}^d \|f\|_{b_{x_j p\theta}^{r_j}(R^d)},$$

and it is called anisotropic Besov space. When $\theta = \infty$, $B_{p\theta}^r(R^d)$ coincides with the Hölder–Nikolski space $H_p^r(R^d)$. We define

$$S_p^r L(R^d) := \left\{f \in L_p(R^d): \|f\|_{L_p^r(R^d)} \leqslant 1\right\},$$
$$S_{p\theta}^r B(R^d) := \left\{f \in L_p(R^d): \|f\|_{B_{p\theta}^r(R^d)} \leqslant 1\right\}.$$

Let $\psi(\cdot)$ is a univariate l-regular wavelet, i.e., $\psi(\cdot)$ satisfy

$$\left|\psi^{(\alpha)}(t)\right| \leqslant c_{m,\alpha}\left(1 + |t|\right)^{-m}, \quad \alpha = 0, 1, \ldots, l, \ m \in N, \ t \in R.$$

The functions

$$\psi_{j,k}(t) = 2^{k/2}\psi\left(2^k t - j\right), \quad j, k \in Z,$$

form an orthogonal basis for $L_2(R)$. For convenience, let $D(R)$ denote the set of dyadic intervals. Each such interval I is of the form $I := I_{j,k} = [j2^{-k}, (j+1)2^{-k}]$, $j, k \in Z$. We define

$$\psi_I := \psi_{j,k}, \quad I = \left[j2^{-k}, (j+1)2^{-k}\right] \in D(R). \tag{1.1}$$

Thus the basis $\{\psi_{j,k}\}_{j,k\in Z}$ is the same as $\{\psi_I\}_{I\in D(R)}$.

For multivariate function space $L_2(R^d)$, we can construct multivariate wavelet basis by taking tensor products of the univariate basis functions. If ψ is a univariate orthogonal wavelet and $d \geqslant 1$, then the functions

$$\psi_{j,k}(t) := \psi_{j_1,k_1}(t_1) \cdots \psi_{j_d,k_d}(t_d), \quad j = (j_1, \ldots, j_d) \in Z^d, \ k = (k_1, \ldots, k_d) \in Z^d, \tag{1.2}$$