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Blaschke-oscillatory equations of the form $f'' + A(z)f = 0$ ☆

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Abstract

We study the zero sequences of the non-trivial solutions of

$$f'' + A(z)f = 0, \tag{†}$$

where $A(z)$ is analytic in the unit disc. We offer several aspects illustrating the fact that it is not so uncommon for these zero sequences to be Blaschke sequences. The typical results can be divided into two categories: (1) We search for conditions on $A(z)$ under which the zero sequences of solutions of (†) are Blaschke sequences. (2) For given Blaschke sequences satisfying certain conditions, we construct an analytic function $A(z)$ (of minimal growth) such that these Blaschke sequences are the zero sequences of certain solutions of (†).

This discussion is a continuation of the recent paper [J. Heittokangas, Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequences as the zeros, Comput. Methods Funct. Theory 5 (2005) 49–63].

Keywords: Zeros of solutions; Zero distribution; Zero sequences; Prescribed zeros; Blaschke sequences

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1. Introduction

In the recent paper [5], the zero distribution of the solutions of

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is analytic in the unit disc $D = \{z\colon\ |z| < 1\}$, is studied from the following points of view:

Problem 1. Find a growth condition for $A(z)$ such that the zero sequence of any non-trivial solution of (1.1) is a Blaschke sequence.

Problem 2. If $\{z_n\}$ is a Blaschke sequence of distinct points in D, then find a function $A(z)$, analytic in D, such that (1.1) possesses a solution having zeros precisely at the points z_n.

An equation of the form (1.1), such that the zero sequence of any non-trivial solution of (1.1) is a Blaschke sequence, is called *Blaschke-oscillatory* in [5]. This new concept generalizes the classical concepts of disconjugate and non-oscillatory equations. An equation of the form (1.1) is called *disconjugate* (respectively *non-oscillatory*), if every non-trivial solution of (1.1) has at most one zero (respectively finitely many zeros) in D. The paper [5] answers to several aspects of the more general problem of characterizing all Blaschke-oscillatory equations of the form (1.1), and the present paper continues this discussion.

The next result is a simple consequence of [11, Theorem 5]. See [5, Section 3] for a further discussion on Problem 1.

Theorem A. *If $A(z)$ is analytic in D satisfying*

$$\int_D \left|A(z)\right|^{\frac{1}{2}}\, d\sigma_z < \infty, \tag{1.2}$$

where $d\sigma_z$ is the Euclidean area measure, then (1.1) *is Blaschke-oscillatory.*

In the converse direction, we obtain

Theorem 1.1. *If $A(z)$ is an analytic function in D such that* (1.1) *is Blaschke-oscillatory, then*

$$\int_D \left|A(z)\right|^{\alpha}\, d\sigma_z < \infty \tag{1.3}$$

holds for every $\alpha \in (0, \frac{1}{2})$.

For the sharpness of Theorem 1.1, we note that there exists Blaschke-oscillatory equations of the form (1.1) in which the coefficient function $A(z)$ does not satisfy (1.2). Indeed, the functions $f_1(z) = (1-z)\exp\bigl(\frac{1+z}{1-z}\bigr)$ and $f_2(z) = (1-z)\exp\bigl(-\frac{1+z}{1-z}\bigr)$, where $z \in D$, are linearly independent solutions of

$$f'' - \frac{4}{(1-z)^4} f = 0. \tag{1.4}$$