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The Hausdorff dimension of a class of recurrent sets ☆

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Abstract

In this paper we obtain a lower bound for the Hausdorff dimension of recurrent sets and, in a general setting, we show that a conjecture of Dekking [F.M. Dekking, Recurrent sets: A fractal formalism, Report 82-32, Technische Hogeschool, Delft, 1982] holds.

Keywords: Hausdorff dimension; Bouligand dimension; Fractal set

1. Introduction

In order to study the formalism of fractals, Dekking [3] has introduced a powerful method of describing and generating fractal sets, which are called recurrent sets. Using this method we can construct almost all well-known sets, for instance, the Peano curve, Cantor sets, and so on. Moreover, in [3], an upper bound for the Hausdorff dimension of a recurrent set is given. When the linear map is a similitude, that is, the modulus of all eigenvalues of the linear map are the same, Dekking [4] conjectured that the Hausdorff dimension of a recurrent set is equal to the upper bound if and only if resolvability holds.

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Later, Bedford [2] and Wen et al. [8], by different methods, proved that the conjecture is true.

It is known that dimension calculations are difficult if the linear map is not a similitude. Bedford [1] and McMullen [6] independently dealt with a simple problem of this sort. Wu [9] also considered the recurrent set if the linear map is not a similitude and obtained the upper and lower bounds for its Hausdorff dimension. In this paper we focus our attention on the case of the non-similitude. Using the methods in [8,9] we will first obtain a lower bound for the Hausdorff dimension of a recurrent set, which is better than that in [9]. Then we show that the conjecture of Dekking is also true in the general setting.

2. Notations and known results

Throughout the paper we denote the Hausdorff, Bouligand, upper Bouligand and lower Bouligand dimensions by $\dim_H$, $\dim_B$, $\overline{\dim}_B$, and $\underline{\dim}_B$, respectively. For a set A, it is obvious that $\dim_H(A) \leqslant \underline{\dim}_B(A) \leqslant \overline{\dim}_B(A)$ (see [5]).

Let S be a finite alphabet of symbols, S^* the free semigroup generated by S, and $\theta : S^* \to S^*$ a semigroup endomorphism. Let $f : S^* \to \mathbb{R}^d$ be a homomorphism, which means that f satisfies $f(uv) = f(u) + f(v)$, for all words $u, v \in S^*$. Assume that L_θ, a representation of θ, is a linear map from $\mathbb{R}^d$ onto itself such that $L_\theta(f(s)) = f(\theta(s))$ for each $s \in S$ and that $K[\cdot] : S^* \to \mathcal{C}(\mathbb{R}^d)$, the family of compact subsets of $\mathbb{R}^d$, is a map satisfying that for all $u, v \in S^*$, $K[uv] = K[u] \cup \{K[v] + f(u)\}$. In this paper, we always suppose that L_θ is expansive, that is, all eigenvalues of L_θ have modulus more than one.

A symbol $s \in S$ is said to be virtual if $K[s] = \emptyset$. The set of virtual symbols is denoted by Q. By [4], we may assume that $\theta Q^* \subset Q^*$ and $\theta(s) \notin Q^*$ for $s \notin Q$. If $s \in E = S \backslash Q$, we say that s is an essential symbol. With these notations, Dekking [3] proved that there exists a non-empty compact set $K_\theta(w)$, called a recurrent set, such that in the Hausdorff metric $L_\theta^{-n} K[\theta^n(w)] \to K_\theta(w)$, as $n \to \infty$, for any word w which contains at least one essential symbol. Moreover, it is shown that $K_\theta(w)$ is independent of the choice of $K[\cdot]$ and so here we take $K[s] = \{\alpha f(s)\colon\ 0 \leqslant \alpha \leqslant 1\}$, for any $s \in S$.

To make an estimation of the growth rate in the number of essential symbols, $|\theta^n(t)|_E$, in $\theta^n(t)$, now we define a non-negative $|E| \times |E|$ matrix $A_E = (a_{st})_{s,t \in E}$ where a_{st} is the number of t in the word $\theta(s)$. We shall assume that θ is essentially mixing, that is, there exists a positive integer m such that $s \in \theta^m(t)$ for all $s, t \in E$. With these notations, A_E is a non-negative matrix and, by Frobewing theorem [7], there exists λ_E being the eigenvalue of A_E with the greatest modulus. Bedford [2] proved that $\dim_H K_\theta(s) = \dim_H K_\theta(t)$, for all $s, t \in E$, and that

$$\lambda_E^n \sim \left|\theta^n(s)\right|_E, \quad n \to \infty. \tag{1}$$

Let $s \in E$, define

$$\begin{aligned}\alpha_0 &= \sup\left\{\alpha\colon\ \liminf_{n\to\infty} \frac{m_d(K[\theta^n(s)])^\varepsilon}{|\theta^n(s)|_E^\alpha} = \infty,\ \text{for some } \varepsilon > 0\right\} \\ &= \inf\left\{\alpha\colon\ \liminf_{n\to\infty} \frac{m_d(K[\theta^n(s)])^\varepsilon}{|\theta^n(s)|_E^\alpha} = 0,\ \text{for some } \varepsilon > 0\right\},\end{aligned} \tag{2}$$