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On semilinear elliptic equations involving concave–convex nonlinearities and sign-changing weight function

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Abstract

In this paper, we study the combined effect of concave and convex nonlinearities on the number of positive solutions for semilinear elliptic equations with a sign-changing weight function. With the help of the Nehari manifold, we prove that there are at least two positive solutions for Eq. $(E_{\lambda,f})$ in bounded domains.

Keywords: Semilinear elliptic equations; Nehari manifold; Concave–convex nonlinearities

1. Introduction

In this paper, we consider the multiplicity results of positive solutions of the following semilinear elliptic equation:

$$\begin{cases} -\Delta u = u^p + \lambda f(x) u^q & \text{in } \Omega, \\ 0 \leqslant u \in H_0^1(\Omega), \end{cases} \tag{$E_{\lambda,f}$}$$

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where Ω is a bounded domain in $\mathbb{R}^N$, $0 < q < 1 < p < 2^*$ ($2^* = \frac{N+2}{N-2}$ if $N \geqslant 3$, $2^* = \infty$ if $N = 2$), $\lambda > 0$ and $f : \overline{\Omega} \to \mathbb{R}$ is a continuous function which change sign in $\overline{\Omega}$. Associated with Eq. $(E_{\lambda,f})$, we consider the energy functional J_λ, for each $u \in H_0^1(\Omega)$,

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx - \frac{\lambda}{q+1}\int_\Omega f(x)|u|^{q+1}\,dx.$$

It is well known that the solutions of Eq. $(E_{\lambda,f})$ are the critical points of the energy functional J_λ (see Rabinowitz [12]).

The fact that the number of positive solutions of Eq. $(E_{\lambda,f})$ is affected by the concave and convex nonlinearities has been the focus of a great deal of research in recent years. If the weight function $f(x) \equiv 1$, the authors Ambrosetti et al. [2] have investigated Eq. $(E_{\lambda,1})$. They found that there exists $\lambda_0 > 0$ such that Eq. $(E_{\lambda,1})$ admits at least two positive solution for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Actually, Adimurthy et al. [1], Damascelli et al. [7], Ouyang and Shi [11], and Tang [16] proved that there exists $\lambda_0 > 0$ such that Eq. $(E_{\lambda,1})$ in the unit ball $B^N(0; 1)$ has exactly two positive solution for $\lambda \in (0, \lambda_0)$, has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$.

The purpose of this paper is to consider the multiplicity of positive solution of Eq. $(E_{\lambda,f})$ for a changing sign potential function $f(x)$. We prove that Eq. $(E_{\lambda,f})$ has at least two positive solutions for λ is sufficiently small.

Theorem 1. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, Eq. $(E_{\lambda,f})$ has at least two positive solutions.*

Among the other interesting problems which are similar of Eq. $(E_{\lambda,f})$ for $q = 0$, Bahri [3], Bahri and Berestycki [4], and Struwe [13] have investigated the following equation:

$$\begin{cases} -\Delta u = |u|^{p-1}u + f(x) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{E_f}$$

where $f \in L^2(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^N$. They found that Eq. (E_f) possesses infinitely many solutions. Furthermore, Cîrstea and Rădulescu [5], Cao and Zhou [6], and Ghergu and Rădulescu [10] have been investigated the analogue Eq. (E_f) in $\mathbb{R}^N$.

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we prove that Eq. $(E_{\lambda,f})$ has at least two positive solutions for λ sufficiently small.

2. Notations and preliminaries

Throughout this section, we denote by S the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$. Now, we consider the Nehari minimization problem: for $\lambda > 0$,

$$\alpha_\lambda(\Omega) = \inf\{J_\lambda(u) \mid u \in \mathbf{M}_\lambda(\Omega)\},$$