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Error bounds in divided difference expansions. A probabilistic perspective ☆

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Abstract

We give error bounds for the remainder term, as well as sharp upper and lower bounds for the leading term, in divided difference expansions. A main feature of this paper is the use of a probabilistic approach in the spirit of Ignatov and Kaishev [Z.G. Ignatov, V.K. Kaishev, B-splines and linear combinations of uniform order statistics, Math. Res. Cent. TSR, 2817, Univ. of Wisconsin, Madison, 1985; Z.G. Ignatov, V.K. Kaishev, A probabilistic interpretation of multivariate B-splines and some applications, Serdica 15 (1989) 91–99] and Karlin et al. [S. Karlin, C.A. Micchelli, Y. Rinott, Multivariate splines: A probabilistic perspective, J. Multivariate Anal. 20 (1986) 69–90] based on the representation of divided differences as mathematical expectations involving linear combinations of uniform order statistics.

Keywords: Divided difference; Modulus of smoothness; Order statistics; Probabilistic Taylor's formula

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1. Introduction

Let n be a positive integer, which we fix from now on, and let $r = 1, \dots, n$. Denote by $\mathbf{F}_r$ the set of all finite ordered sequences of real numbers $\mathbf{x} := (x_0, \dots, x_n)$ having $r+1$ distinct knots among $x_0, \dots, x_n$, that is,

$$(x_0, \dots, x_n) = (\underbrace{\tilde{x}_0, \dots, \tilde{x}_0}_{m_0}, \dots, \underbrace{\tilde{x}_r, \dots, \tilde{x}_r}_{m_r}). \tag{1}$$

If $\mathbf{x} \in \mathbf{F}_r$, we write $h_i := x_i - x_{i-1}$, $i = 1, \dots, n$, $\tilde{h}_i := \tilde{x}_i - \tilde{x}_{i-1}$, $i = 1, \dots, r$, $h := \max(h_1, \dots, h_n)$ and $\tilde{h} := \min(\tilde{h}_1, \dots, \tilde{h}_r)$. We define the sets

$$\mathbf{F} := \bigcup_{r=1}^{n} \mathbf{F}_r \quad \text{and} \quad \mathbf{F}_r(a) := \left\{ \mathbf{x} \in \mathbf{F}_r \colon \frac{\tilde{h}}{h} \geqslant a \right\}, \quad 0 < a \leqslant 1.$$

If $\mathbf{x} \in \mathbf{F}$, we denote by $I := [x_0, x_n]$ and by $C^m(I)$ the set of m times differentiable functions f defined on I such that $f^{(m)}$ is continuous, $m = 1, 2, \dots$. For any $\mathbf{x} \in \mathbf{F}_r$, we denote by $[x_0, \dots, x_n]f$ the nth divided difference of the function f.

For any $\mathbf{x} \in \mathbf{F}$ and $f \in C^p(I)$, $p = n+1, n+2, \dots$, Floater [6] has shown divided difference expansions of the form

$$[x_0, \dots, x_n]f = \sum_{k=n}^{p-1} c_k \frac{f^{(k)}(x)}{k!} + R_p, \quad x \in I, \tag{2}$$

where each coefficient c_k is expressible in terms of symmetric polynomials and the remainder term R_p is bounded in terms of the maximum grid spacing h. As an important special case, this author shows the following error bound for any $f \in C^{n+2}(I)$:

$$\left| n![x_0, \dots, x_n]f - f^{(n)}(\bar{x}) \right| \leqslant \frac{n}{24} h^2 \left\| f^{(n+2)} \right\|, \tag{3}$$

where $\|\cdot\|$ stands for the usual sup-norm on I and $\bar{x} := (n+1)^{-1}(x_0 + \cdots + x_n)$ is the arithmetic mean of the knots. The constant $n/24$ in (3) is the best possible, in the sense that it is attained in the case of equidistant simple knots. As Floater [6] points out, the error bound in (3) is interesting in the design of finite difference schemes over nonuniform grids, which are useful in the numerical solutions to differential equations.

In this paper, we complete some of the results given in [6]. On the one hand, we provide general bounds for the remainder term in (2) under slightly weaker differentiability assumptions on the function under consideration (see Theorem 1 in Section 3). On the other, we give a detailed analysis of the leading term implicitly contained in (3), under the same differentiability conditions as those in [6] (see Section 4). Exact values, as well as sharp upper and lower bounds for the corresponding leading coefficient are provided in Propositions 3–5 in Section 4. It turns out that such bounds heavily depend on the number $r+1$ of distinct knots $\tilde{x}_i$ and their associated multiplicities m_i, and therefore are given on each of the sets $\mathbf{F}_r$ and $\mathbf{F}_r(a)$.

The main tool in this paper is the following probabilistic representation of divided differences (see Section 2):

$$n![x_0, \dots, x_n]f = Ef^{(n)}\big(Z(\mathbf{x})\big), \quad f \in C^n(I),\ \mathbf{x} \in \mathbf{F}, \tag{4}$$