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Note

A characterization of inverse Radon transform on the Laguerre hypergroup ☆

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Abstract

Let $\mathbb{K} = [0, \infty) \times \mathbb{R}$ be the Laguerre hypergroup which is the fundamental manifold of the radial function space for the Heisenberg group. In this note we give another characterization for a subspace of $\mathcal{S}(\mathbb{K})$ (Schwartz space) such that the Radon transform R_α on $\mathbb{K}$ is a bijection. We show that this characterization is equivalent to that in [M.M. Nessibi, K. Trimèche, Inversion of the Radon transform on the Laguerre hypergroup by using generalized wavelets, J. Math. Anal. Appl. 208 (1997) 337–363]. In addition, we establish an inversion formula of the Radon transform R_α in the weak sense.

Keywords: Radon transform; Wavelet transform; Laguerre hypergroup

1. Introduction

In the past decade research on Radon transform has made considerable progress due to its wide applications to partial differential equations, X-ray technology, radio astronomy and so on. For the basic theory and further results of Radon transform we refer readers to the book [1] by S. Helgason and the references therein. The combination of Radon

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transform and wavelet transform has proved to be very useful both in pure mathematics and its applications. Recently, various authors deal with the inversion formula of Radon transform by using inverse wavelet transform (see [2–5]). When one considers the problems of radial function on the Heisenberg group, the fundamental manifold is the Laguerre hypergroup $\mathbb{K} = [0, \infty) \times \mathbb{R}$. In [3], M.M. Nessibi and K. Trimèche defined the Radon transform R_α on $\mathbb{K}$, and characterized a subspace $\mathcal{S}_{*,2}(\mathbb{K})$ of Schwartz space on which the Radon transform R_α is a bijection. Moreover, they obtained an inversion formula of the Radon transform on $\mathcal{S}_{*,2}(\mathbb{K})$ by use of the generalized wavelet transform. In this note we give another characterization which seems to be natural, we show that this characterization is equivalent to that in [3]. In addition, we obtain an inversion of the Radon transform on $\mathbb{K}$ in the weak sense.

Let $\mathbb{K} = [0, \infty) \times \mathbb{R}$, $\alpha \geqslant 0$, the Radon transform R_α on $\mathbb{K}$ is given by

$$R_\alpha f(x,t) = \frac{2\pi^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty T^{(\alpha)}_{(x,t)} f(y,0) y^{2\alpha+1}\, dy, \tag{1}$$

where $T^{(\alpha)}_{(x,t)}$, $(x,t) \in \mathbb{K}$, are the generalized translation operators on $\mathbb{K}$ defined by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f\big(\sqrt{x^2+y^2+2xy\cos\theta}, s+t+xy\sin\theta\big)\, d\theta, \\ \quad \text{if } \alpha = 0, \\ \frac{\alpha}{\pi} \int_0^{2\pi} \int_0^1 f\big(\sqrt{x^2+y^2+2xyr\cos\theta}, s+t+xyr\sin\theta\big) \\ \quad \times r(1-r^2)^{\alpha-1}\, dr\, d\theta, \quad \text{if } \alpha > 0. \end{cases}$$

Let $dm_\alpha(x,t)$ be the positive measure on $\mathbb{K}$ given by $dm_\alpha(x,t) = \frac{1}{\pi\Gamma(\alpha+1)} x^{2\alpha+1}\, dx\, dt$. Then $L^2_\alpha(\mathbb{K})$ denotes the space of measurable functions on $\mathbb{K}$ such that

$$\|f\|_{\alpha,2} = \left(\int_{\mathbb{K}} \big|f(x,t)\big|^2\, dm_\alpha(x,t) \right)^{1/2} < +\infty. \tag{2}$$

The convolution product of f and g on $\mathbb{K}$ is defined by

$$(f * g)(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s) g(y,-s)\, dm_\alpha(y,s). \tag{3}$$

For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, $(x,t) \in \mathbb{K}$, we set $\varphi_{\lambda,m}(x,t) = e^{i\lambda t} \mathcal{L}^{(\alpha)}_m(|\lambda| x^2)$, where $\mathcal{L}^{(\alpha)}_m$ denotes the Laguerre function defined on $[0, +\infty)$ by $\mathcal{L}^{(\alpha)}_m(x) = e^{-x/2} L^{(\alpha)}_m(x)/L^{(\alpha)}_m(0)$, $L^{(\alpha)}_m$ being the Laguerre polynomial of degree m and order α. The generalized Fourier transform on $\mathbb{K}$ is defined by

$$\hat{f}(\lambda, m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t) f(x,t)\, dm_\alpha(x,t). \tag{4}$$

Let $L^2_\alpha(\mathbb{R} \times \mathbb{N})$ be the space of measurable functions $g : \mathbb{R} \times \mathbb{N} \mapsto \mathbb{C}$ such that