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Existence of periodic solutions for second-order differential equations with singularities and the strong force condition

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Abstract

We prove the existence of periodic solutions for the equation

$$u'' + f(u)u' + g(t,u) = e(t), \tag{1}$$

where the nonlinearity g has a repulsive singularity at the origin. In previous papers dealing with this kind of problem it is usually assumed a nonintegrability condition on g near the origin. We provide a weaker condition that substitutes the nonintegrability of g. If $f \equiv 0$ the existence of subharmonic solutions is proved utilizing a variational method and when $f \neq 0$ we prove the existence of a periodic solution using topological degree theory.

Keywords: Periodic solutions; Subharmonics; Repulsive singularities; Truncation

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1. Introduction

This paper deals with the problem of finding periodic solutions, and in particular subharmonic solutions for the forced oscillator with differential equation

$$u'' + f(u)u' + g(t,u) = e(t), \tag{2}$$

where for $a = -\infty$ or $a = 0$ we suppose that $f :]a, +\infty[\to \mathbf{R}$ is continuous and $g : \mathbf{R} \times]a, +\infty[\to \mathbf{R}$ is T-periodic in its first variable and of L^1-Carathéodory type, that is: $g(\cdot, x)$ is measurable for each $x \in]a, +\infty[$, $g(t, \cdot)$ is continuous for a.e. $t \in \mathbf{R}$, and for every $a < s < S$ there exists $h_{s,S} \in L^1_{\mathrm{loc}}(\mathbf{R})$ in such a way that $|g(t,u)| \leqslant h_{s,S}(t)$ for all $u \in [s, S]$ and a.e. $t \in \mathbf{R}$. Moreover $e : \mathbf{R} \to \mathbf{R}$ is locally integrable and T-periodic.

We are mainly interested in singular nonlinearities, that is, we suppose that g becomes unbounded near the origin. This situation was extensively studied, both, the case with friction (see [7,10–12]) and without (see [1,3,9]). This study was done independently for the case of an attractive force (if $g(t,x) \to +\infty$ when $x \to 0^+$) and the repulsive one (if $g(t,x) \to -\infty$ when $x \to 0^+$).

Lazer and Solimini, in a paper of 1987 (see [8]), prove the existence of a periodic solution of the equation

$$u'' + g(u) = e(t) \tag{3}$$

with a repulsive force. They assume a "strong force condition"

$$\int_0^1 g(u)\,du = -\infty \tag{4}$$

and show that there are nonlinearities g, not verifying the strong force condition, for which there are functions e such that Eq. (3) does not have solutions. Since then, the condition (4) has been considered as necessary for the existence of periodic solutions of (2) and is commonly assumed (see [1,3,5,7,9–12]). The condition (4) has as a consequence that the energy of a solution passing near the origin is arbitrarily large. Keeping in mind the same idea we will show that this condition can be weakened in such a way that it permits some kind of unification with the nonsingular case. Throughout this paper we will assume the existence of a continuous function $l :]a, +\infty[\to \mathbf{R}$ such that $g(t,u) < l(u)$, for all $u \in]a, +\infty[$ and a.e. $t \in [0, T]$, verifying the following condition:

(L_1) There exists a sequence $]r_n, r_n'[\subset]a, 1[$ such that $l(u) < \bar{e}$ for all

$$u \in \bigcup_{n \in \mathbf{N}}]r_n, r_n'[\quad \text{and} \quad \int_{r_n}^{r_n'} l(u)\,du \to -\infty.$$

Notice that if $a = 0$ this condition is equivalent to (4) when l has an upper bound near the origin. So we are thinking of examples such as

$$l(u) = \frac{1}{u^3} \sin\left(\frac{1}{u}\right)$$

that cannot be classified either as repulsive neither as attractive.