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On the leading eigenvalue of neutron transport models

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Abstract

We give variational characterizations of the leading eigenvalue of neutron transport-like operators. The proofs rely on sub- and super-eigenvalues. Various bounds of the leading eigenvalue are derived.

Keywords: Positivity; Leading eigenvalue; Sub-eigenvalue; Super-eigenvalue; Transport equations

1. Introduction

This paper provides a new approach of the leading eigenvalue for neutron transport-like equations. The so-called time eigenvalue of the fundamental mode (i.e. the leading eigenvalue) of neutron transport operators plays a basic role in nuclear reactor theory, e.g., in pulsed experiments [6, Chapter 5] or in the stochastic description of neutron chain fissions [3]. This eigenvalue or, more generally, the peripheral spectrum of such operators is strongly related to their positivity properties (in the lattice sense); see [17] and references therein. In the same spirit, positivity plays an essential role in reactor criticality; see [14] and references therein. We refer to [10, Chapter 5] and references therein for the known results on the leading eigenvalue of neutron transport operators. Motivated by transport theory, the present paper is devoted to *variational characterizations* of the lead-

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ing eigenvalue for a class of perturbed operators of the form $A = T + K$ where T is an unbounded operator with a *positive resolvent* and K is a bounded *positive* operator. If we denote respectively by $s(T)$ and $s(A)$ the spectral bound of T and A and if some power of $(\lambda - T)^{-1}K$ is compact $(\lambda > s(T))$, then it is known that $s(A)$ is the leading eigenvalue of A once $s(T) < s(A)$ [16]. Here, this leading eigenvalue is handled by means of sub-eigenvalues or super-eigenvalues. Roughly speaking, we prove that $\lambda \in]s(T), s(A)[$ if and only if λ is a *sub-eigenvalue*, i.e. there exists a nonnegative (non-trivial) φ such that $A\varphi \geqslant \lambda\varphi$. We show also that $\lambda \in]s(A), \infty[$ if and only if λ is a *super-eigenvalue*, i.e. there exists a nonnegative (non-trivial) φ such that $A\phi \leqslant \lambda\phi$. It follows that $s(A)$ can be characterized as the supremum of sub-eigenvalues or the infimum of super-eigenvalues. This provides us with *max–inf* and *min–sup* principles for the leading eigenvalue. This first part of our work, of more functional analytic character, is in the spirit of I. Marek [9] who deals, in particular, with variational characterizations of spectral radius of certain positive operators. In the second part, devoted specifically to neutron transport, we show how to derive in a systematic manner, from the above (abstract) variational principles, upper and lower bounds of the leading eigenvalue in terms of various physical parameters. This paper resumes some results from a longer preliminary version [12] containing additional results and references. We present now our general framework. Let $\Omega \subset R^N$ be a smooth and bounded open set and let μ be a positive Radon measure on R^N with support V. We refer to V as the velocity space. We assume in this paper that V is *bounded away from zero*, i.e. $0 \notin V$. We refer to [12] for the case $0 \in V$. Let T be the advection operator in $L^p(\Omega \times V) := L^p(\Omega \times V; dx\, d\mu(v))$ $(1 \leqslant p < \infty)$

$$T\varphi = -v \cdot \frac{\partial \varphi}{\partial x} - \sigma(x, v)\varphi(x, v), \quad \varphi \in D(T)$$

with domain

$$W^p_{0-} = \left\{ \varphi \in L^p(\Omega \times V);\ v \cdot \frac{\partial \varphi}{\partial x} \in L^p(\Omega \times V),\ \varphi = 0 \text{ on } \Gamma_- \right\}$$

where $\Gamma_- := \{(x, v) \in \partial\Omega \times V;\ \ v \cdot n(x) < 0\}$ and $n(x)$ is the outward unit vector at $x \in \partial\Omega$. The real and bounded measurable function $\sigma(\cdot,\cdot)$ is the collision frequency while the scattering (or collision) operator is

$$K : \varphi \in L^p(\Omega \times V) \to \int_V k(x, v, v')\varphi(x, v')\, d\mu(v') \in L^p(\Omega \times V).$$

Finally, the neutron transport operator is given by

$$A : \varphi \in W^p_{0-} \to -v \cdot \frac{\partial \varphi}{\partial x} - \sigma(x, v)\varphi(x, v) + \int_V k(x, v, v')\varphi(x, v')\, d\mu(v')$$

with the same domain as the advection operator T. The cross sections $\sigma(\cdot,\cdot)$ and $k(\cdot,\cdot,\cdot)$ are *nonnegative* in accordance with the physical theory. The spectral bound of T, $s(T) = \sup\{\operatorname{Re}\lambda; \lambda \in \sigma(T)\}$, is characterized in full generality in [18]: $s(T) = -\lambda^*$ where

$$\lambda^* = \lim_{t\to\infty} \inf_{\{(x,v)\in\Omega\times V;\ t<\tau(x,-v)\}} t^{-1} \int_0^t \sigma(x + sv, v)\, ds$$