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Improvement of Newman inequality ☆

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Abstract

Let $a = \exp(-1/\sqrt{n})$. Newman inequality is

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} < e^{-\sqrt{n}}, \quad \forall n \geqslant 5.$$

We prove in this paper that

$$\prod_{k=1}^{s-1} \frac{1-a^k}{1+a^k} = n^{\frac{1}{4}} e^{-\frac{\pi^2}{4}\sqrt{n}+\mathcal{O}(1)}, \quad \forall s \geqslant n,$$

which will be applied to improve the estimate concerning the approximation of $|x|$ by using Newman's construction.

Keywords: Newman inequality; Rational approximation

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1. Introduction

Let $a = \exp(-1/\sqrt{n})$. In 1964 Newman established the following well-known inequality:

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} < e^{-\sqrt{n}}, \quad \forall n \geqslant 5. \tag{1.1}$$

Using this inequality Newman proved

$$\max_{|x|\leqslant 1} \big| |x| - N_n(x) \big| \leqslant 3e^{-\sqrt{n}}, \quad \forall n \geqslant 5, \tag{1.2}$$

where the rational functions $N_n(x)$ are given by

$$N_n(x) = x\frac{P(x) - P(-x)}{P(x) + P(-x)} \quad \text{and} \quad P(x) = \prod_{k=1}^{n-1} \big(a^k + x\big).$$

Because of the simple construction of N_n Newman's approach was used widely to construct interesting rational functions in approximation theory (see [1–4,6] and the papers cited there).

Recently we find out that the right-hand side of (1.1) can be replaced by a smaller number, thus (see [7]), there holds

$$\prod_{k=1}^{n-1} \frac{1-a^k}{1+a^k} < C' e^{-1.3\sqrt{n}}, \tag{1.3}$$

where $C' > 0$ is an absolute constant. Using this inequality, we obtain the asymptotic express of $\max_{|x|\leqslant 1} ||x| - N_n(x)|$ (see [7]):

$$\max_{|x|\leqslant 1} \big| |x| - N_n(x) \big| = \frac{A}{\sqrt{n}} e^{-\sqrt{n}} + \mathcal{O}\left(\frac{1}{n} e^{-\sqrt{n}}\right), \tag{1.4}$$

where $A = \max_{0\leqslant t\leqslant\infty} t/(1+e^t)$. Therefore, the exact approximation rate for $|x|$ by $N_n(x)$ is $e^{-\sqrt{n}}/\sqrt{n}$.

Comparing (1.1) and (1.3) with Newman's approach it is natural to ask what is the exact order in (1.3) and whether can we modify this term to improve the approximation rate of $|x|$. The aim of this paper is to answer these questions. For this goal let us denote

$$\Delta_s = \prod_{k=1}^{s-1} \frac{1-a^k}{1+a^k}, \quad \forall s = n, n+1, \ldots.$$

Let further

$$N_{n,s}(x) = x\frac{P_s(x) - P_s(-x)}{P_s(x) + P_s(-x)} \quad \text{and} \quad P_s(x) = \prod_{k=1}^{s-1} \big(a^k + x\big).$$

The main result of this paper is