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Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces ☆

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Abstract

In this paper, we established two strong convergence theorems for a multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense(asymptotically quasi-nonexpansive, respectively) in Banach spaces. Our results extend and improve the recent ones announced by Xu and Noor [B.L. Xu, M.A. Noor, Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 (2002) 444–453], Cho, Zhou and Guo [Y.J. Cho, H. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (2004) 707–717], and many others.

Keywords: Asymptotically nonexpansive in the intermediate sense; Asymptotically quasi-nonexpansive mappings; Completely continuous; Uniformly convex; Uniformly L-Lipschitzian

1. Introduction

Let C be a subset of real normed linear space X. A mapping $T: C \to C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \leqslant (1 + r_n)\|x - y\|, \quad \forall n \geqslant 1.$$

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If $r_n \equiv 0$, then T is known as a *nonexpansive mapping*. T is called *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\|T^n x - T^n y\right\| - \|x-y\|\right) \leqslant 0.$$

T is said to be *asymptotically quasi-nonexpansive mapping*, if there exists a sequence $\{r_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for all $x \in C$, $p \in F(T)$,

$$\left\|T^n x - p\right\| \leqslant (1+r_n)\|x-p\|,$$

for all $n \geqslant 1$, where $F(T)$ denotes the set of fixed points of T, i.e., $F(T) = \{x \in C\colon\ Tx = x\}$. T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \leqslant L\|x-y\|,$$

for all $n \geqslant 1$ and $x, y \in C$.

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense, asymptotically quasi-nonexpansive mapping and L-Lipschitzian mapping. But the convergence does not hold such as in the following example:

Example 1.1. (See [9]) Let $X = \mathbb{R}$, $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is asymptotically nonexpansive in the intermediate sense. *It is well known* [8] *that $T^n x \to 0$ uniformly, but is not a Lipschitzian mapping so that it is not asymptotically nonexpansive mapping*.

Fixed-point iterations process for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequations; see [6–18]. In 2000, Noor [13] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge, Nguyen and Strodiot [5] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [3] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences.

Recently, Xu and Noor [19] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho, Zhou and Guo [2] extended the work of Xu and Noor to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [18] gave weak and strong convergence theorems for a new three-step iterative scheme of asymptotically nonexpansive mappings. Inspired and motivated by these