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Positive solution for m-point boundary value problems of fourth order

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Abstract

In this paper, we consider the existence of positive solutions to the fourth order boundary value problem

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t,u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases}$$

where $\alpha, \beta \in R$, $\xi_i \in (0,1)$, $a_i, b_i \in [0,\infty)$ for $i \in \{1,2,\ldots,m-2\}$ are given constants satisfying some suitable conditions. The proofs are based on the fixed point index theorem in cones.

Keywords: Multipoint boundary value problems; Positive solution; Fixed point index; Cones; Eigenvalue

1. Introduction

Multipoint boundary value problems (BVPs) for ordinary differential equations arise in a variety of areas of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multipoint BVP in [4]; also, many problems in the theory of elastic stability can be handled by multipoint problems in [5].

In [6], Il'in and Moiseev studied the existence of solutions for a linear multipoint BVP. Motivated, Gupta [7] studied certain three-point BVPs for nonlinear ordinary differential equations.

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Since then, more general nonlinear multipoint BVPs have been studied by several authors. We refer the reader to [2,3,7] for some references. When it comes to fourth order BVP, only two-point situation is often seen, here we just refer to [8,9]. Recently, Liu [8] has obtained some existence results for

$$\begin{cases} u'''' = f(t, u, u''), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1a}$$

under the condition that f is either superlinear or sublinear.

In 2003, Li [1] studied the existence of positive solutions for fourth order BVP without bending term but with two parameters

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t, u), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases} \tag{1.1b}$$

under the assumptions:

(H1) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous;
(H2) $\alpha, \beta \in R$ and $\alpha < 2\pi^2$, $\beta \geqslant -\alpha^2/4$, $\alpha/\pi^2 + \beta/\pi^4 < 1$.

He established the following result for (1.1b).

Theorem 1.1. [1] *Assume* (H1) *and* (H2) *hold. Then in each of the following cases*:

(i) $\overline{f}_0 < \pi^4 - \alpha\pi^2 - \beta$, $\underline{f}_\infty > \pi^4 - \alpha\pi^2 - \beta$,
(ii) $\underline{f}_0 > \pi^4 - \alpha\pi^2 - \beta$, $\overline{f}_\infty < \pi^4 - \alpha\pi^2 - \beta$,

the BVP (1.1b) *has at least one positive solution, where*

$$\underline{f}_0 = \liminf_{u\to 0+} \min_{t\in[0,1]} \big(f(t, u)/u\big), \qquad \overline{f}_0 = \limsup_{u\to 0+} \max_{t\in[0,1]} \big(f(t, u)/u\big),$$

$$\underline{f}_\infty = \liminf_{u\to +\infty} \min_{t\in[0,1]} \big(f(t, u)/u\big), \qquad \overline{f}_\infty = \limsup_{u\to +\infty} \max_{t\in[0,1]} \big(f(t, u)/u\big).$$

In this paper, we are interested in the existence of a positive solution for the more general fourth order m-point BVP,

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t,u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} a_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} b_i u''(\xi_i), \end{cases} \tag{1.1}$$

where $\alpha, \beta \in R, \xi_i \in (0, 1), a_i, b_i \in [0, \infty)$ for $i \in \{1, 2, \dots, m-2\}$ are given constants. It is clear that when $a_i = b_i = 0$, then (1.1) is reduced to (1.1b). To deal with (1.1), we give an integral equation which is equivalent to (1.1). It is naturally expected that an integral equation equivalent to

$$\begin{cases} u'''' + \alpha u'' - \beta u = f(t,u), \quad 0 < t < 1, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \\ u''(0) = \sum_{i=1}^{m-2} c_i u''(\xi_i), \quad u''(1) = \sum_{i=1}^{m-2} d_i u''(\xi_i) \end{cases}$$

holds true. It at last fails when we attempt on it. Hence we just consider (1.1).