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Existence for a semilinear sixth-order ODE ☆

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Dedicated to Professor Dan Pascali on the occasion of his 70th birthday

Abstract

In this paper we study the existence and multiplicity of nontrivial solutions for a boundary value problem associated with a semilinear sixth-order ordinary differential equation arising in the study of spatial patterns. Our treatment is based on variational tools, including two Brezis–Nirenberg's linking theorems.

Keywords: Semilinear sixth-order ODE; Variational method; Brezis–Nirenberg's linking theorems

1. Introduction

In this paper, we study the existence and multiplicity of the solutions of the following boundary value problem, say (P):

$$u^{vi} + Au^{iv} + Bu'' + Cu = f(x,u), \quad 0 < x < L, \tag{1}$$

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$$u(0) = u(L) = u''(0) = u''(L) = u^{iv}(0) = u^{iv}(L) = 0, \tag{2}$$

where A, B and C are some given real constants and $f(x,u)$ is a continuous function on $\mathbb{R}^2$, whose potential satisfies some suitable assumptions.

The problem is motivated by the study of the formation of the spatial periodic patterns in bistable systems. In investigating such spatial patterns, a key role is played by a model equation, which is simpler than the full equation describing the process. Recently, interest has turned to fourth-order parabolic differential equations, involving bistable dynamics, such as the extended Fisher–Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \quad \gamma > 0,$$

proposed by Coullet, Elphick and Repaux in 1987 as well as by Dee and Van Saarlos in 1988. Another well-known equation of this type is the Swift–Hohenberg (SH) equation

$$\frac{\partial u}{\partial t} = \alpha u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3, \quad \alpha > 0,$$

proposed in 1977. While both the EFK equation and the SH equation involve fourth-order spatial derivatives, certain phase-field models lead to even higher-order spatial gradients. We mention the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + u - u^3,$$

studied by Gardner and Jones [6] as well as by Caginalp and Fife [1].

If f is an even $2L$-periodic function with respect to x, and odd with respect to u, the $2L$-periodic extension $\bar{u}$ of the odd extension of the solution u of problem (P) to the interval $[-L, L]$ yields a $2L$-periodic solution of Eq. (1).

The existence of periodic solutions of both the EFK equation and the SH equation was studied by Peletier and Troy [4], Peletier and Rottschäfer [5], Tersian and Chaparova [7] and other authors.

In this paper we introduce an extended class for the nonlinear term $f(x,u)$ which includes the typical example $f(x,u) = b(x)u|u|^{p-2}$, where $p > 2$ and $b(x)$ is a continuous positive function.

We suppose $f(x,0) = 0$, $\forall x \in \mathbb{R}$, and the potential

$$F(x,u) = \int_0^u f(x,s)\,ds$$

satisfies the following assumptions:

(H_1) $\frac{F(x,u)}{u^2} \to +\infty$ as $|u| \to +\infty$, uniformly with respect to x in bounded intervals, and
(H_2) $0 \leqslant F(x,u) = o(u^2)$ as $u \to 0$, uniformly with respect to x in bounded intervals.