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Absolute equal distribution of the eigenvalues of discrete Sturm–Liouville problems

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Abstract

We consider the asymptotic relationship as $n \to \infty$ between the eigenvalues $\lambda_{1n} \leqslant \cdots \leqslant \lambda_{nn}$ and $\mu_{1n} \leqslant \cdots \leqslant \mu_{nn}$ of the Sturm–Liouville problems defined for $n \geqslant 2k+1$ by

$$\sum_{\ell=0}^{k} (-1)^\ell \Delta^\ell \big(r_{\ell n}(i-\ell) \Delta^\ell x_{i-\ell} \big) = \lambda \phi_{in} x_i, \quad 1 \leqslant i \leqslant n,$$

and

$$\sum_{\ell=0}^{k} (-1)^\ell \Delta^\ell \big(s_{\ell n}(i-\ell) \Delta^\ell x_{i-\ell} \big) = \mu \psi_{in} x_i, \quad 1 \leqslant i \leqslant n,$$

where $x_i = 0$ if $-k+1 \leqslant i \leqslant 0$ or $n+1 \leqslant i \leqslant n+k$, all quantities are real, and $\phi_{in}, \psi_{in} > 0$, $1 \leqslant i \leqslant n$, $n \geqslant 2k+1$. We give conditions implying that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \big| F(\lambda_{in}) - F(\mu_{in}) \big| = 0$$

for all $F \in C(-\infty, \infty)$ such that $\lim_{x\to-\infty} F(x)$ and $\lim_{x\to\infty} F(x)$ exist (finite).

Keywords: Absolute equal distribution; Boundary conditions; Eigenvalue; Sturm–Liouville

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1. Introduction

We consider the asymptotic relationship as $n \to \infty$ between the eigenvalues of the Sturm–Liouville problems defined for $n \geqslant 2k+1$ by

$$\sum_{\ell=0}^{k}(-1)^{\ell}\Delta^{\ell}\big(r_{\ell n}(i-\ell)\Delta^{\ell}x_{i-\ell}\big)=\lambda\phi_{in}x_i, \quad 1 \leqslant i \leqslant n, \tag{1}$$

and

$$\sum_{\ell=0}^{k}(-1)^{\ell}\Delta^{\ell}\big(s_{\ell n}(i-\ell)\Delta^{\ell}x_{i-\ell}\big)=\mu\psi_{in}x_i, \quad 1 \leqslant i \leqslant n, \tag{2}$$

with boundary conditions

$$x_i = 0 \quad \text{if } -k+1 \leqslant i \leqslant 0 \text{ or } n+1 \leqslant i \leqslant n+k. \tag{3}$$

We assume throughout that all quantities are real and $\phi_{in}, \psi_{in} > 0$, $1 \leqslant i \leqslant n$, $n \geqslant 2k+1$. If $\{d_n\}$ is a sequence of positive numbers, we write $a_n = o(d_n)$ to indicate that $\lim_{n\to\infty} a_n/d_n = 0$ and $a_n = O(d_n)$ to indicate that $\limsup_{n\to\infty} |a_n|/d_n < \infty$.

Definition 1. Let $\#S$ denote the cardinality of a set S. We say that a family of ordered sets $\{\{c_{in}\}_{i=i_0}^{n}\}_{n=2k+1}^{\infty}$ is essentially null if

$$\#\big\{i \mid i_0 \leqslant i \leqslant n \text{ and } |c_{in}| > \epsilon\big\} = o(n)$$

for every $\epsilon > 0$. We say that $\{\{c_{in}\}_{i=i_0}^{n}\}_{n=2k+1}^{\infty}$ is essentially bounded if there is an $M \in [0,\infty)$ such that

$$\#\big\{i \mid i_0 \leqslant i \leqslant n \text{ and } |c_{in}| > M\big\} = o(n).$$

This definition is closely related to [3, Definitions 4.2–4.4].

Let $\mathcal{F}$ be the set of all $F \in C(-\infty,\infty)$ such that

$$\lim_{x\to-\infty} F(x) \quad \text{and} \quad \lim_{x\to\infty} F(x) \quad \text{exist (finite).} \tag{4}$$

Let $\lambda_{1n} \leqslant \lambda_{2n} \leqslant \cdots \leqslant \lambda_{nn}$ and $\mu_{1n} \leqslant \mu_{2n} \leqslant \cdots \leqslant \mu_{nn}$ be the eigenvalues of (1), (3) and (2), (3), respectively. The following theorems are proved in Section 2.

Theorem 1. *If* $\{\{\phi_{in}\}_{i=1}^{n}\}_{n=2k+1}^{\infty}$, $\{\{1/\phi_{in}\}_{i=1}^{n}\}_{n=2k+1}^{\infty}$, *and*

$$\big\{\big\{r_{\ell n}(i)\big\}_{i=-\ell+1}^{n}\big\}_{n=2k+1}^{\infty}, \quad 0 \leqslant \ell \leqslant k,$$

are essentially bounded and

$$\big\{\{\phi_{in}-\psi_{in}\}_{i=1}^{n}\big\}_{n=2k+1}^{\infty} \quad \text{and} \quad \big\{\big\{r_{\ell n}(i)-s_{\ell n}(i)\big\}_{i=-\ell+1}^{n}\big\}_{n=2k+1}^{\infty}, \quad 0 \leqslant \ell \leqslant k,$$

are essentially null, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\big|F(\lambda_{in})-F(\mu_{in})\big| = 0 \quad \text{for all } F \in \mathcal{F}. \tag{5}$$