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Viscosity approximation methods for nonexpansive nonself-mappings ☆

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Abstract

Let E a real reflexive Banach space which admits a weakly sequentially continuous duality mapping from E to E^*, and K be a closed convex subset of E which is also a sunny nonexpansive retract of E, and $T : K \to E$ be nonexpansive mappings satisfying the weakly inward condition and $F(T) \neq \emptyset$, and $f : K \to K$ be a fixed contractive mapping. The implicit iterative sequence $\{x_t\}$ is defined by for $t \in (0, 1)$

$$x_t = P\big(tf(x_t) + (1-t)Tx_t\big).$$

The explicit iterative sequence $\{x_n\}$ is given by

$$x_{n+1} = P\big(\alpha_n f(x_n) + (1-\alpha_n)Tx_n\big),$$

where $\alpha_n \in (0, 1)$ and P is sunny nonexpansive retraction of E onto K. We prove that $\{x_t\}$ strongly converges to a fixed point of T as $t \to 0$, and $\{x_n\}$ strongly converges to a fixed point of T as α_n satisfying appropriate conditions. The results presented extend and improve the corresponding results of [H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291].

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1. Introduction and preliminaries

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x)=\{f\in E^*:\ \langle x,f\rangle=\|x\|\|f\|,\ \|x\|=\|f\|\},\quad \forall x\in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j, and denote $F(T)=\{x\in E:\ Tx=x\}$. When $\{x_n\}$ is a sequence in E, then $x_n\to x$ (respectively $x_n\rightharpoonup x$, $x_n\overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x. In Banach space E, the following result (*the Subdifferential Inequality*) is well known [1,9]: $\forall x,y\in E$, $\forall j(x+y)\in J(x+y)$, $\forall j(x)\in J(x)$,

$$\|x\|^2+2\langle y,j(x)\rangle\leqslant\|x+y\|^2\leqslant\|x\|^2+2\langle y,j(x+y)\rangle. \tag{1.1}$$

Let E be a real Banach space and T a mapping with domain $D(T)$ and range $R(T)$ in E. T is called *nonexpansive* (respectively *contractive*) if for any $x,y\in D(T)$, such that

$$\|Tx-Ty\|\leqslant\|x-y\|$$

(respectively $\|Tx-Ty\|\leqslant\beta\|x-y\|$ for some $0<\beta<1$). If I denotes the identity operator and T is a contractive mapping, the following inequality holds [3]:

$$\langle (I-T)x-(I-T)y,j(x-y)\rangle\geqslant(1-\beta)\|x-y\|^2. \tag{1.2}$$

Let K be a closed convex subset of a uniformly smooth Banach space E, $T:K\to K$ a nonexpansive mapping with $F(T)\neq\emptyset$, $f:K\to K$ a contraction. Then for any $t\in(0,1)$, the mapping

$$T_t^f: x\mapsto tf(x)+(1-t)Tx$$

is also contraction. Let x_t denote the unique fixed point of T_t^f. In [3], H.K. Xu proved that as $t\downarrow 0$, $\{x_t\}$ converges to a fixed point p of T that is the unique solution of the variational inequality

$$\langle (I-f)u,j(u-p)\rangle\leqslant 0\quad\text{for all } p\in F(T).$$

H.K. Xu also propose the following explicit iterative process $\{x_n\}$ given by

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Tx_n,$$

and prove that the explicit iterative process $\{x_n\}$ converges to a fixed point p of T.

In this paper, our purpose is to prove that in reflexive Banach space E which admits a weakly sequentially continuous duality mapping from E to E^*, for nonexpansive nonself-mapping T, both $\{x_t\}$ defined by (1.3) and $\{x_n\}$ defined by (1.4) strongly converges to a fixed point of T, which generalizes and improves several recent results. Particularly, it extends and improves Theorems 4.1 and 4.2 of [3].