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# Perron's method for quasilinear hyperbolic systems: Part I

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**Abstract**

We define a notion of viscosity solution (sub-, supersolution) for these systems, prove a comparison principle and we prove existence of viscosity solutions using a Perron like method. In Part I, we do all the above except prove existence using the Perron method.


*Keywords:* Hyperbolic systems; Viscosity solutions

## 1. Introduction

In several previous papers [1–3] the author proved a comparison principle for a semilinear second order wave equation, defined a notion of viscosity solution for such equations, and showed that Perron's method extended to these hyperbolic equations to prove existence of viscosity solutions of the Cauchy problem for these equations.

Such equations can be written as first order symmetric hyperbolic systems [4,5]. In this paper, we prove comparison theorems for sub- and supersolutions of first order symmetric hyperbolic systems, define a notion of viscosity solution for these systems, and extend Perron's method of upper and lower envelopes to prove the existence of continuous viscosity solutions for the long time and eternal Cauchy problem for such systems.

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Our motivation is provided by the fact that this will yield a new numerical method to solve such systems which include most of the important equations of mathematical physics and, in particular, the long term Cauchy problem for Einstein field equations, which [5–7] can be shown to be equivalent to solving such a system.

Indeed, in a second paper, we use the results of this paper, and a gauge independent method, due to Deturk [7], of showing the above equivalence, to show the existence of long term, possible only continuous, viscosity solutions of the Einstein–Cauchy problem.

We also note that the Navier–Stokes equations (for supersonic flow) can be put in this form, and we have results in progress on the general long term existence for continuous viscosity solutions of these equations that use the results of this paper.

## 2. Quasilinear first order symmetric hyperbolic systems

For the convenience of the reader, we recall the definition of a first order, quasilinear, symmetric hyperbolic system in a slab domain $D_T$ of $R^+ \times R^n$. More details may be found in [5,8].

**Definition 1.** Let $n, N \in Z^+$. Let $D_T := [0, T] \times R^n$.

Let $u : R^+ \times R^n \to R^N$.

Let $A^0 : R^+ \times R^n \times R^N \to \mathrm{Sym}^+(N, N)$ be $C^\infty$ (in fixed coordinates, $A^0$ is an $N \times N$ symmetric matrix), and let $A^0$ be positive definite, that is: all of its eigenvalues are strictly bigger than zero.

Let $i = 1, 2, \ldots, n$. For each $i$, let

$$A^i : R^+ \times R^n \times R^N \to \mathrm{Sym}(N, N)$$

be $C^\infty$ and in $C^{2+\epsilon_0}$ $\exists\epsilon_0 \in (0, 1)$, where we require a uniform Hölder constant for the whole domain $D_T$ (in a fixed coordinates each $A^i$ is a symmetric matrix). Let $\mathrm{Sup}_{D_T}\{|A^i|, |D_{x,t}A^i|, |D^2_{x,t}A^i|\} < \infty$. Let $\partial_i u$ and $\partial_0 u$ exist in $D^0_T$.

**Definition 2.** Let $B : R^+ \times R^n \to L(N, N)$ be $C^\infty$. Let all the off diagonal terms of $B$ be nonnegative. We call such $A^i$ admissible.

**Definition 3.** Let $f : R^+ \times R^n \to R^N$ be $C^\infty$, let $f(t, x) \geqslant 0$, and let $\lim_{|x|\to\infty} f(t, x) = 0$. Let $f \in H^{1,2}(R^n, R^N)$, where we mean that one weak derivative is square integrable. Let $f \in C^{\epsilon_0}(D_T, R^N)$, with $\epsilon_0$ as above. We call such $f$ admissible.

**Definition 4.** We say that $u$ satisfies a *first order quasilinear symmetric hyperbolic system* in $D^0_T$ iff: in $D^0_T$:

$$Lu := A^0(t, x, u)\partial_t u + A^i(t, x, u)\partial_i u - B(t, x)u - f(t, x) = 0.$$

(Here repeated indices are summed.) Also we require $A^i$ and $f$ to be admissible.

Note that a.n.l.o.g., by standard methods [5,8], we can assume that $A^0 = I$. We do this from now on, for simplicity of presentation.