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Original article

The second Darboux problem for the wave equation with integral nonlinearity

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Abstract

For a one-dimensional wave equation with integral nonlinearity, the second Darboux problem is considered for which the questions on the existence and uniqueness of a global solution are investigated.

Keywords: Darboux problem; Integral nonlinearity; Wave equation; Global solvability

1. Statement of the problem

In a plane of independent variables x and t we consider the wave equation with integral nonlinearity of the type

$$L_\lambda u := u_{tt} - u_{xx} + \lambda g\left(x, t, u, \int_{\alpha(t)}^{\beta(t)} u(x,t)dx\right) = f(x,t), \tag{1.1}$$

where $\lambda \neq 0$ is the given real constant; g, α, β and f are the given and u is an unknown real functions of their arguments.

By $D_T := \{(x,t) \in \mathbb{R}^2 : -\widetilde{k}_2 t < x < \widetilde{k}_1 t, 0 < t < T; 0 < \widetilde{k}_i := \text{const} < 1, i = 1,2\}$ we denote a triangular domain lying inside of a characteristic angle $\Lambda := \{(x,t) \in \mathbb{R}^2 : t > |x|\}$ and bounded by the segments $\widetilde{\gamma}_{1,T} : x = \widetilde{k}_1 t, 0 \le t \le T$, $\widetilde{\gamma}_{2,T} : x = -\widetilde{k}_2 t, 0 \le t \le T$ and $\widetilde{\gamma}_{3,T} : t = T, -\widetilde{k}_2 T \le x \le \widetilde{k}_1 T$. For $T = +\infty$, $D_\infty := \{(x,t) \in \mathbb{R}^2 : -\widetilde{k}_2 t < x < \widetilde{k}_1 t, 0 < t < +\infty\}$ (Fig. 1.1).

For Eq. (1.1), let us consider the second Darboux problem on finding in the domain D_T a solution $u(x,t)$ of the above equation by the boundary conditions (see e.g., [1, p. 107]; [2, p. 228])

$$u|_{\widetilde{\gamma}_{i,T}} = 0, \quad i = 1,2. \tag{1.2}$$

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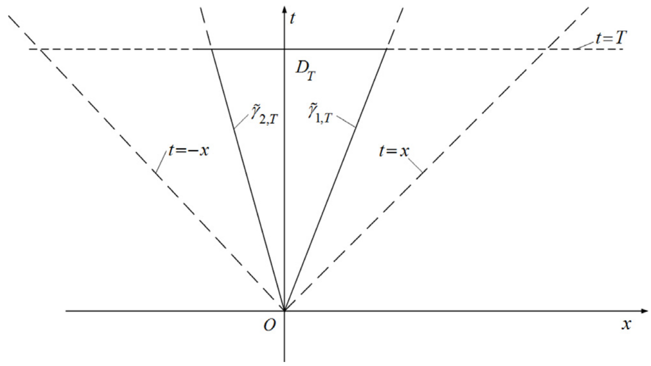

Fig. 1.1.

Below, when investigating problem (1.1), (1.2) it will be assumed that

$$-\widetilde{k}_2\, t \le \alpha(t) < \beta(t) \le \widetilde{k}_1\, t, \quad 0 < t < \infty. \tag{1.3}$$

For linear hyperbolic equations of second order with one spatial variable, a great number of works were devoted to the questions of the well-posedness of the Darboux problem (see, e.g., [2,3] and references therein). As it turned out, the presence of a weak nonlinearity in the equation affects the correctness of formulation even in the case of the first Darboux problem (see, e.g., [4–10]). Note that hyperbolic equations with nonlocal nonlinearities of type (1.1) have been considered in many works (see, e.g., [11–14] and references therein). In the present work it is shown that under definite conditions on the growth of nonlinear function $g = g(x, t, s_1, s_2)$ with respect to the variables s_1, s_2 the second Darboux problem (1.1), (1.2) is globally solvable.

Definition 1.1. Let $\alpha, \beta \in C([0, T])$, $g \in C(\overline{D}_T \times \mathbb{R}^2)$, $f \in C(\overline{D}_T)$. The function u is said to be a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ such that $u_n \to u$ and $L_\lambda u_n \to f$ in the space $C(\overline{D}_T)$, as $n \to \infty$, where $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T) := \{v \in C^2(\overline{D}_T) : v|_{\Gamma_T} = 0\}$, $\Gamma_T := \widetilde{\gamma}_{1,T} \cup \widetilde{\gamma}_{2,T}$.

Remark 1.1. Note that two different approximations with given properties define the same function in Definition 1.1. Obviously, the classical solution of problem (1.1), (1.2) from the space $\overset{\circ}{C}{}^2(\overline{D}_T, \Gamma_T)$ is a strong generalized solution of that problem of the class C in the domain D_T. In its turn, if a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T belongs to the space $C^2(\overline{D}_T)$, then it will be a classical solution of that problem, as well.

Definition 1.2. Let $\alpha, \beta \in C([0, \infty))$, $g \in C(\overline{D}_\infty \times \mathbb{R}^2)$, $f \in C(\overline{D}_\infty)$. We say that problem (1.1), (1.2) is globally solvable in the class C, if for any finite $T > 0$, this problem has a strong generalized solution of the class C in the domain D_T.

2. An a priori estimate of solution of problem (1.1), (1.2)

Let us consider the following condition imposed on the function g:

$$\big|g(x, t, s_1, s_2)\big| \le a + b|s_1| + c|s_2|, \quad (x, t, s_1, s_2) \in \overline{D}_T \times \mathbb{R}^2, \tag{2.1}$$

where $a, b, c = \text{const} \ge 0$.

Lemma 2.1. *Let the condition (2.1) be fulfilled. Then for a strong generalized solution of problem (1.1), (1.2) of the class C in the domain D_T the following a priori estimate*

$$\|u\|_{C(\overline{D}_T)} \le c_1 \|f\|_{C(\overline{D}_T)} + c_2 \tag{2.2}$$

with nonnegative constants c_i, $i = 1, 2$, independent of u and f, is valid.