Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 170 (2016) 395–401
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Finite difference scheme for one nonlinear parabolic integro-differential equation

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Received 22 July 2016; received in revised form 19 September 2016; accepted 26 September 2016
Available online 14 October 2016

Abstract

Initial–boundary value problem with mixed boundary conditions for one nonlinear parabolic integro-differential equation is considered. The model is based on Maxwell system describing the process of the penetration of a electromagnetic field into a substance. Unique solvability and asymptotic behavior of solution are fixed. Main attention is paid to the convergence of the finite difference scheme. More wide cases of nonlinearity that already were studied are investigated.

MSC: 65N06; 45K05; 35K55

Keywords: Nonlinear parabolic integro-differential equation; Initial–boundary value problem; Mixed boundary conditions; Finite difference scheme; Convergence

1. Introduction

Integro-differential equations of parabolic type arise in the study of various problems (see, for example, [1–5] and references therein). One such model is obtained at mathematical modeling of processes of electromagnetic field penetration in the substance. It is shown that in the quasi-stationary approximation the corresponding system of Maxwell equations [6] can be rewritten in the following form [7]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rot H|^2\, d\tau\right) rot H\right], \tag{1.1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that integro-differential models of (1.1) type are complex and still yields to the investigation only for special cases (see, for example, [3,7–20] and references therein).

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Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.09.006

Study of the models of type (1.1) has begun in the work [7]. In particular, for the case $a(S) = 1 + S$ the theorems of existence of solution of the first boundary value problem for scalar and one-dimensional space case and uniqueness for more general cases are proved in this work. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \le 1$ is studied in [9]. Investigations for multi-dimensional space cases at first are carried out in the work [10]. Multidimensional space cases are also discussed in the following works [14,18].

Asymptotic behavior as $t \to \infty$ of solutions of initial–boundary value problems for (1.1) type models is studied in the works [3,11,14–16] and in a number of other works as well. In these works main attentions, are paid to one-dimensional analogs.

Interest to above-mentioned integro-differential model is more and more arising and initial–boundary value problems with different kinds of boundary and initial conditions are considered. Particular attention should be paid to construction of numerical solutions and to their importance for integro-differential models. Finite element analogs and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (1.1) type one-dimensional integro-differential models are studied in [12,16,20–22] and in the other works as well (see [3] and references therein).

Our main aim is to study finite difference scheme for numerical solution of initial–boundary value problem with mixed boundary conditions for the one-component and one-dimensional analog of (1.1) system. Attention is paid to the investigation of more wide cases of nonlinearity than already were studied.

This article is organized as follows. In Section 2 the formulation of the problem and unique solvability and asymptotic behavior of solution are fixed. Main attention is paid to construction and investigation of finite difference scheme in Section 3. We conclude the paper with some discussions in Section 4.

2. Formulation of the problem. Unique solvability and asymptotic behavior of solution

If the magnetic field has the form $H = (0, 0, U)$, $U = U(x, t)$, then from (1.1) we obtain the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \tag{2.1}$$

where

$$S(x, t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau. \tag{2.2}$$

In the domain $[0, 1] \times [0, \infty)$ let us consider the following initial–boundary value problem for (2.1), (2.2):

$$U(0, t) = \left.\frac{\partial U(x, t)}{\partial x}\right|_{x=1} = 0, \tag{2.3}$$

$$U(x, 0) = U_0(x), \tag{2.4}$$

where U_0 is a given function.

The study of unique solvability and long-time behavior of solution of the problem (2.1)–(2.4) is actual.

The following statement [13] shows the exponential stabilization of the solution of problem (2.1)–(2.4) in the norm of the space $C^1(0, 1)$.

Theorem 2.1. *If* $a(S) = (1 + S)^p$, $0 < p \le 1$ *and* $U_0 \in H^3(0, 1)$, $U_0(0) = \left.\frac{dU_0(x)}{dx}\right|_{x=1} = 0$, *then for the solution of problem* (2.1)–(2.4) *the following estimates hold as* $t \to \infty$*:*

$$\left|\frac{\partial U(x, t)}{\partial x}\right| \le C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial U(x, t)}{\partial t}\right| \le C \exp\left(-\frac{t}{2}\right),$$

uniformly in x *on* $[0, 1]$.

Using the compactness method, a modified version of the Galerkin method [5,23] the unique solvability can be proven.