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Original article

On the Opial type criterion for the well-posedness of the Cauchy problem for linear systems of ordinary differential equations

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Abstract

There are obtained necessary and sufficient conditions for the well-posedness of the Cauchy problem for the systems of linear ordinary differential equations, analogous to the sufficient condition by Z. Opial for the problem one. Moreover, there are given the efficient sufficient conditions for the problem one.

Keywords: Linear systems of ordinary differential equations; The Cauchy problem; Well-posedness; The Opial type condition; Necessary and sufficient conditions; Efficient sufficient conditions

1. Statement of the problem and basic notation

Let $P_0 \in L_{loc}(I, \mathbb{R}^{n\times n})$, $q_0 \in L_{loc}(I, \mathbb{R}^n)$ and $t_0 \in I$, where I is an arbitrary interval from $\mathbb{R}$ non-degenerated in the point. Let x_0 be a unique solution of the Cauchy problem

$$\frac{dx}{dt} = \mathcal{P}_0(t)\,x + q_0(t), \tag{1.1}$$

$$x(t_0) = c_0, \tag{1.2}$$

where $c_0 \in \mathbb{R}^n$ is a constant vector.

Consider sequences of matrix- and vector-functions $P_k \in L_{loc}(I, \mathbb{R}^{n\times n})$ $(k = 1, 2, \dots)$ and $q_k \in L_{loc}(I, \mathbb{R}^n)$ $(k = 1, 2, \dots)$, respectively; sequence of points t_k $(k = 1, 2, \dots)$ and sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$.

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In [1–8] (see, also the references therein), the sufficient conditions are given such that a sequence of unique solutions x_k $(k = 1, 2, \ldots)$ of the Cauchy problems

$$\frac{dx}{dt} = \mathcal{P}_k(t)\,x + q_k(t), \tag{1.1$_k$}$$

$$x(t_k) = c_k \tag{1.2$_k$}$$

$(k = 1, 2, \ldots)$ satisfy the condition

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on } I. \tag{1.3}$$

In the present paper necessary and sufficient conditions are established for the sequence of the Cauchy problems (1.1$_k$), (1.2$_k$) $(k = 1, 2, \ldots)$ to have the above-mentioned property. The obtained criterion are based on the concept by Z. Opial, concerning to the sufficient condition considered in [8], and it differs from analogous one given in [1].

The Opial type sufficient conditions are investigated in [5] for the well-posedness problem of the Cauchy problem for linear functional-differential equations.

In the paper the use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$; $[a, b]$ and $]a, b[(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

I is an arbitrary, non-degenerated in the point, finite or infinite interval from $\mathbb{R}$.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1,\ldots,m} \sum_{i=1}^{n} |x_{ij}|.$$

$O_{n\times m}$ is the zero $n \times m$-matrix.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$; o_n is the zero n-vector.

$\mathbb{R}^{n\times n}$ is the space of all real quadratic $n \times n$-matrices $X = (x_{ij})_{i,j=1}^n$;

I_n is the identity $n \times n$-matrix; $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$; δ_{ij} is the Kronecker symbol, i.e. $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j\,(i, j = 1, \ldots)$;

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are, respectively, the matrix inverse to X and the determinant of X; $\operatorname{diag}X = \operatorname{diag}(x_{11}, \ldots, x_{nn})$ is the diagonal matrix corresponding to X.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

We say that the matrix-function $X \in L_{loc}(I, \mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition if for every $\tau \in I$ the following condition holds

$$X(t)\int_\tau^t X(\tau)\,d\tau = \int_\tau^t X(\tau)d\tau \cdot X(t) \quad \text{for a. a. } t \in I.$$

$\mathop{\mathrm{V}}\limits_a^b(X)$ is the sum total variation of the components $x_{ij}\,(i = 1, \ldots, n;\ j = 1, \ldots, m)$ of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$; $\mathop{\mathrm{V}}\limits_b^a(X) = -\mathop{\mathrm{V}}\limits_a^b(X)$;

$\mathop{\mathrm{V}}\limits_I(X) = \lim_{a\to\alpha+,b\to\beta-} \mathop{\mathrm{V}}\limits_a^b(X)$, where $\alpha = \inf I$ and $\beta = \sup I$.

$C(I; \mathbb{R}^{m\times n})$ is a space of continuous and bounded matrix-functions $X : I \to \mathbb{R}^{m\times n}$ with the norm

$$\|X\|_c = \sup\{\|X(t)\| : t \in I\};$$

$C(I; D)$, where $D \subset \mathbb{R}^{m\times n}$, is the set of continuous and bounded matrix-functions $X : I \to D$;

$C_{loc}(I; D)$ is the set of continuous matrix-functions $X : I \to D$;

$\widetilde{C}(I; D)$ is the set of absolutely continuous matrix-functions $X : I \to D$;

$\widetilde{C}_{loc}(I; D)$ is the set of matrix-functions $X : I \to D$ which are absolutely continuous on the every closed interval $[a, b]$ from I.

$L(I; D)$, where $D \subset \mathbb{R}^{m\times n}$, is the set of matrix-functions $X : I \to D$ whose components are Lebesgue-integrable;