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Original article

On the cardinal number of the family of all invariant extensions of a nonzero σ-finite invariant measure

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Abstract

It is shown that, for any nonzero σ-finite translation invariant (translation quasi-invariant) measure μ on the real line $\mathbf{R}$, the cardinality of the family of all translation invariant (translation quasi-invariant) measures on $\mathbf{R}$ extending μ is greater than or equal to 2^{ω_1}, where ω_1 denotes the first uncountable cardinal number. Some related results are also considered.

Keywords: Invariant measure; Quasi-invariant measure; Metrical transitivity; Nonseparable extension of measure; Ulam's transfinite matrix

Let E be a base (ground) set and let G be a group of transformations of E. The pair (E, G) is usually called a space equipped with a transformation group.

A measure μ defined on some G-invariant σ-algebra of subsets of E is called quasi-invariant with respect to G (briefly, G-quasi-invariant) if, for any μ-measurable set X and for any transformation g from G, the relation

$$\mu(X) = 0 \quad \Leftrightarrow \quad \mu(g(X)) = 0$$

holds true. Moreover, if the equality $\mu(g(X)) = \mu(X)$ is valid for any μ-measurable X and for any g from G, then μ is called an invariant measure with respect to G (briefly, G-invariant measure).

According to these definitions, the triplet of the form (E, G, μ) determines the structure of an invariant (quasi-invariant) measure on E.

Suppose that μ is a nonzero σ-finite G-invariant (G-quasi-invariant) measure on E. It is known that if a group G is uncountable and acts freely in E, then there always exist subsets of E nonmeasurable with respect to μ (see [1]; cf. also [2]). So the domain of μ differs from the family of all subsets of E, i.e., $\mathrm{dom}(\mu) \neq \mathcal{P}(E)$. In this connection, the natural question arises whether there exists a G-invariant (G-quasi-invariant) measure μ' on E strongly extending μ. This question was studied for various types of spaces (E, G, μ). Undoubtedly, the most interesting case for classical Real Analysis is when E coincides with the n-dimensional Euclidean space $\mathbf{R}^n$, a group G is a subgroup of the group of all isometric transformations of $\mathbf{R}^n$, and μ is a G-invariant extension of the standard n-dimensional Lebesgue measure λ_n on $\mathbf{R}^n$ (see, for instance, [3–8]).

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Another important case is when $E = \Gamma$, where Γ is an uncountable σ-compact locally compact topological group, Γ coincides with the group of all left (right) translations of Γ, and μ is a G-invariant extension of the left (right) Haar measure on Γ (cf. [9,5,10,11]).

A more general form of the above question is as follows. For a given space (E, G, μ), denote by $\mathcal{M}_G(\mu)$ the family of all measures on E extending μ and invariant (quasi-invariant) with respect to G. It is natural to try to evaluate the cardinality of $\mathcal{M}_G(\mu)$ in terms of $\text{card}(E)$ and $\text{card}(G)$. In the present paper, we will be dealing with this problem for the case when E coincides with the real line $\mathbf{R}$ and G is the group of all translations of $\mathbf{R}$. Notice that the method applied in our further considerations is primarily taken from [6].

Below, we will use the following standard notation:

$X \triangle Y =$ the symmetric difference of two sets X and Y;
$\omega =$ the least infinite cardinal (ordinal) number;
$\omega_1 =$ the least uncountable cardinal (ordinal) number;
$\mathbf{c} =$ the cardinality of the continuum.

Let μ be a measure defined on some σ-algebra of subsets of E (here μ is not assumed to be invariant or quasi-invariant under a nontrivial group of transformations of E). The Hilbert space of all square μ-integrable real-valued functions on E is usually denoted by the symbol $L_2(\mu)$. If $L_2(\mu)$ is a separable Hilbert space, then μ is called a separable measure. Otherwise, μ is called a nonseparable measure.

Treating the real line $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$ of all rational numbers and keeping in mind the existence of a Hamel basis in $\mathbf{R}$, it is not difficult to show that the additive group $(\mathbf{R}, +)$ admits a representation in the form

$$\mathbf{R} = G + H \quad (G \cap H = \{0\}),$$

where G and H are some two subgroups of $(\mathbf{R}, +)$ and

$$\text{card}(G) = \omega_1, \quad \text{card}(H) \leq \mathbf{c}.$$

We denote by $\mathcal{I}$ the σ-ideal generated by all those subsets X of $\mathbf{R}$ which are representable in the form $X = Y + H$, where $Y \subset G$ and $\text{card}(Y) \leq \omega$.

It can readily be seen that $\mathcal{I}$ is a translation invariant σ-ideal of sets in $\mathbf{R}$.

We begin with the following auxiliary statement.

Lemma 1. *There exists a partition $\{X_\xi : \xi < \omega_1\}$ of $\mathbf{R}$ satisfying these two relations:*

(1) *for any ordinal $\xi < \omega_1$, the set X_ξ belongs to the σ-ideal $\mathcal{I}$;*
(2) *for each subset Ξ of ω_1 and for any $r \in \mathbf{R}$, the relation*

$$(\cup\{X_\xi : \xi \in \Xi\}) \triangle (r + \cup\{X_\xi : \xi \in \Xi\}) \in \mathcal{I}$$

holds true, i.e., the set $\cup\{X_\xi : \xi \in \Xi\}$ is $\mathcal{I}$-almost translation invariant in $\mathbf{R}$.

The proof of this lemma is given in [6].

By combining Lemma 1 with the well-known $(\omega \times \omega_1)$-matrix of Ulam (see, e.g., [12]), the next auxiliary statement can be deduced.

Lemma 2. *Let $\{X_\xi : \xi < \omega_1\}$ be a partition of $\mathbf{R}$ described in Lemma 1 and let μ be a nonzero σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$.*

There exists a disjoint family $\{\Xi_j : j \in J\}$ of subsets of ω_1 such that:

(1) $\text{card}(J) = \omega_1$*;*
(2) *for each index $j \in J$, the set $Z_j = \cup\{X_\xi : \xi \in \Xi_j\}$ is nonmeasurable with respect to μ (where $\{X_\xi : \xi < \omega_1\}$ is a partition of $\mathbf{R}$ described in Lemma 1);*
(3) $\mu_*(\cup\{Z_j : j \in J\}) = 0$ *(where the symbol μ_* denotes the inner measure associated with μ).*

Notice that the proof of Lemma 2 is similar to the argument presented in [6] (cf. also [7]).

Lemma 3. *Let μ be a σ-finite translation invariant (translation quasi-invariant) measure on $\mathbf{R}$. There exists a measure μ' on $\mathbf{R}$ such that:*