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Original article

On sets of singular rotations for translation invariant bases

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Abstract

The following problem is studied: For a summable function f, what kind may be a set of all rotations γ for which $\int f$ is not differentiable with respect to the γ-rotation of the given basis B? In particular, for translation invariant bases on the plane, the topological structure of possible sets of singular rotations is found.

Keywords: Differentiation basis; Translation invariant basis; Rotation; Integral

1. Definitions and notation

A mapping B defined on $\mathbb{R}^n$ is said to be a *differentiation basis* if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of bounded measurable sets with positive measure and containing x, such that there exists a sequence $R_k \in B(x)(k \in \mathbb{N})$ with $\lim_{k\to\infty} \operatorname{diam} R_k = 0$.

For $f \in L(\mathbb{R}^n)$, the numbers

$$\overline{D}_B\left(\int f, x\right) = \overline{\lim_{\substack{R\in B(x)\\ \operatorname{diam} R\to 0}}} \frac{1}{|R|}\int_R f \quad \text{and} \quad \underline{D}_B\left(\int f, x\right) = \underline{\lim_{\substack{R\in B(x)\\ \operatorname{diam} R\to 0}}} \frac{1}{|R|}\int_R f$$

are called *the upper and the lower derivatives,* respectively, *of the integral of f at a point x*. If the upper and the lower derivative coincide, then their combined value is called the *derivative of $\int f$ at the point x* and we denote it by $D_B(\int f, x)$. We say that the *basis B differentiates $\int f$* (or $\int f$ is differentiable with respect to B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions X, we say that B differentiates X.

Denote by $\mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ the basis of intervals, i.e., the basis for which $\mathbf{I}(x)$ $(x \in \mathbb{R}^n)$ consists of all open n-dimensional intervals containing x. Note that the differentiation with respect to $\mathbf{I}$ is called *strong differentiation*.

For the basis B, by F_B we denote the class of all functions $f \in L(\mathbb{R}^n)$ whose integrals are differentiable with respect to B.

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The basis B is called *translation invariant* (briefly, TI-basis) if $B(x) = \{x + R : R \in B(0)\}$ for every $x \in \mathbb{R}^n$.

Denote by Γ_n the family of all rotations in the space $\mathbb{R}^n$.

Let B be the basis in $\mathbb{R}^n$ and $\gamma \in \Gamma_n$. The γ*-rotated basis* B is defined as follows:

$$B(\gamma)(x) = \{x + \gamma(R - x) : R \in B(x)\} \quad (x \in \mathbb{R}^n).$$

The set of two-dimensional rotations Γ_2 can be identified with the circumference $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, if to the rotation γ we put into correspondence the complex number $z(\gamma)$ from $\mathbb{T}$, the argument of which is equal to the value of the angle by which the rotation about the origin takes place in the positive direction under the action of γ.

The distance $d(\gamma, \sigma)$ between the points $\gamma, \sigma \in \Gamma_2$ is assumed to be equal to the length of the smallest arch of the circumference $\mathbb{T}$ connecting the points $z(\gamma)$ and $z(\sigma)$.

Let B and H be bases in $\mathbb{R}^n$ and $E \subset \Gamma_n$. Let us call E a $W_{B,H}$*-set* ($W^+_{B,H}$*-set*), if there exists a function $f \in L(\mathbb{R}^n)$ ($f \in L(\mathbb{R}^n)$, $f \geq 0$) such that: (1) $f \notin F_{B(\gamma)}$ for every $\gamma \in E$ and (2) $f \in F_{H(\gamma)}$ for every $\gamma \notin E$.

Let B and H be bases in $\mathbb{R}^n$ and $E \subset \Gamma_n$. Let us call E an $R_{B,H}$*-set* ($R^+_{B,H}$*-set*), if there exists a function $f \in L(\mathbb{R}^n)$ ($f \in L(\mathbb{R}^n)$, $f \geq 0$) such that: (1) $\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty$ almost everywhere for every $\gamma \in E$ and (2) $f \in F_{H(\gamma)}$ for every $\gamma \notin E$.

When $B = H$, we will use the terms W_B (W^+_B, R_B, R^+_B)-set.

Remark 1. It is clear that:

(1) each $W^+_{B,H}$ ($R^+_{B,H}$)-set is $W_{B,H}$ ($R_{B,H}$)-set;

(2) if $B \subset H$, then each W_B (W^+_B, R_B, R^+_B)-set is $W_{B,H}$ ($W^+_{B,H}$, $R_{B,H}$, $R^+_{B,H}$)-set.

The definitions of $R_{\mathbf{I}(\mathbb{R}^2)}$, $R^+_{\mathbf{I}(\mathbb{R}^2)}$ and $W_{\mathbf{I}(\mathbb{R}^2)}$-sets were introduced in [1,2] and [3], respectively.

2. Results

Singularities of an integral of a fixed summable function with respect to the collection of rotated bases $B(\gamma)$ were studied by various authors (see [1–9]). In particular, in [1] and [3], one can find the proof of the following results dealing with the topological structure of $R_{\mathbf{I}(\mathbb{R}^2)}$-sets and $W_{\mathbf{I}(\mathbb{R}^2)}$-sets, respectively.

Theorem A. *Each* $R_{\mathbf{I}(\mathbb{R}^2)}$*-set has* G_δ *type.*

Theorem B. *Each* $W_{\mathbf{I}(\mathbb{R}^2)}$*-set has* $G_{\delta\sigma}$ *type.*

The following generalizations of Theorems A and B are true.

Theorem 1. *For an arbitrary translation invariant basis* B *in* $\mathbb{R}^2$*, each* W_B*-set has* $G_{\delta\sigma}$ *type.*

Theorem 2. *For an arbitrary translation invariant basis* B *in* $\mathbb{R}^2$*, each* R_B*-set has* G_δ *type.*

Theorems 1 and 2 were announced in [10].

We will also prove the following result.

Theorem 3. *For arbitrary bases* B *and* H *in* $\mathbb{R}^2$ *not more than a countable union of* $R_{B,H}$*-sets* ($R^+_{B,H}$*-sets*) *is* $W_{B,H}$*-set* ($W^+_{B,H}$*-set*).

Proof of Theorem 1. Let $f \in L(\mathbb{R}^2)$. We have to prove that the set

$$W_B(f) = \{\gamma \in \Gamma_2 : f \notin F_{B(\gamma)}\}$$

is of $G_{\delta\sigma}$ type.

Without loss of generality, let us assume that f is finite everywhere and $\operatorname{supp} f \subset (0, 1)^n$.