Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 170 (2016) 47–55
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Long-time behavior of solution and semi-discrete scheme for one nonlinear parabolic integro-differential equation

Temur Jangveladze*

Ilia Vekua Institute of Applied Mathematics of Ivane Javakhishvili Tbilisi State University, 2, University st., Tbilisi 0186, Georgia
Georgian Technical University, 77, Kostava Ave., Tbilisi 0175, Georgia

Available online 13 January 2016

Abstract

Long-time behavior of solution and semi-discrete scheme for one nonlinear parabolic integro-differential equation are studied. Initial–boundary value problem with mixed boundary conditions are considered. Attention is paid to the investigation of more wide cases of nonlinearity than already were studied. Considered model is based on Maxwell's system describing the process of the penetration of a magnetic field into a substance.

Keywords: Nonlinear parabolic integro-differential equation; Initial–boundary value problem; Mixed boundary conditions; Long-time behavior; Semi-discrete scheme

1. Introduction

Integro-differential equations of parabolic type arise in the study of various problems in physics, chemistry, technology, economics, etc. (see, for example, [1–3] and references therein). One such model is obtained by mathematical modeling of processes of electromagnetic field penetration in the substance. In the quasi-stationary approximation the corresponding system of Maxwell's equations has the form [4]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} = \nu_m \left(\operatorname{rot} H\right)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, ν_m characterizes the electro-conductivity of the substance. Eq. (1.1) describes the process of diffusion of the magnetic field and Eq. (1.2)—change of the temperature at the expense of Joule's heating. If ν_m depends on temperature θ, i.e., $\nu_m = \nu_m(\theta)$, then the system (1.1),

* Correspondence to: Ilia Vekua Institute of Applied Mathematics of Ivane Javakhishvili Tbilisi State University, 2, University st., Tbilisi 0186, Georgia.

E-mail address: tjangv@yahoo.com.

Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2015.12.002

(1.2) can be rewritten in the following form [5]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\left(\int_0^t \left|\operatorname{rot} H\right|^2 d\tau\right)\operatorname{rot} H\right], \tag{1.3}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that integro-differential parabolic models of (1.3) type are complex and still yields to the investigation only for special cases (see, for example, [6–9,5,10–16] and references therein).

Study of the models of type (1.3) has begun in the work [5]. In particular, for the case $a(S) = 1 + S$ the theorems of existence of solution of the first boundary value problem for scalar and one-dimensional space case and uniqueness for more general cases are proved in this work. One-dimensional scalar variant for the case $a(S) = (1 + S)^p$, $0 < p \le 1$ is studied in [7]. Investigations for multi-dimensional space cases at first are carried out in the work [8]. Multidimensional space cases are also discussed in the following works [11,14].

Asymptotic behavior as $t \to \infty$ of solutions of initial–boundary value problems for (1.3) type models are studied in the work [9,11–13] and in a number of other works as well. In these works main attention is paid to one-dimensional analogs.

Interest to above-mentioned integro-differential models is more and more arising and initial–boundary value problems with different kinds of boundary and initial conditions are considered. Particular attention should be paid to construction of numerical solutions and to their importance for integro-differential models. Finite element analogs and Galerkin method algorithm as well as settling of semi-discrete and finite difference schemes for (1.3) type one-dimensional integro-differential models are studied in [10,13,17,16] and in the other works as well.

Our aim is to study long-time behavior of solution and semi-discrete scheme for numerical solution of initial–boundary value problem with mixed boundary condition for the one-dimensional (1.3) equation. Attention is paid to the investigation of more wide cases of nonlinearity than already were studied.

This article is organized as follows. In Section 2 the formulation of the problem and asymptotic behavior of solution is studied. Main attention is paid to construction and investigation of semi-discrete scheme in Section 3. We conclude the paper with some discussion of future research in this area in Section 4.

2. Long-time behavior of solution

If the magnetic field has the form $H = (0, 0, U)$, $U = U(x, t)$, then from (1.3) we obtain the following nonlinear integro-differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \tag{2.1}$$

where

$$S(x, t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau. \tag{2.2}$$

In the domain $(0, 1) \times (0, \infty)$ let us consider the following initial–boundary value problem for (2.1), (2.2):

$$U(0, t) = \left.\frac{\partial U(x, t)}{\partial x}\right|_{x=1} = 0,$$
$$U(x, 0) = U_0(x), \tag{2.3}$$

where U_0 is a given function.

The study of long-time behavior of solution of the problem (2.1)–(2.3) is actual.

The following statement shows the exponential stabilization of the solution of problem (2.1)–(2.3) in the norm of the Sobolev space $H^1(0, 1)$.

Theorem 1. *If $a(S) = (1 + S)^p$, $0 < p \le 1$ and $U_0 \in H^2(0, 1)$, $U_0(0) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = 0$, then for the solution of problem (2.1)–(2.3) the following estimate holds as $t \to \infty$*

$$\left\|\frac{\partial U}{\partial x}\right\| + \left\|\frac{\partial U}{\partial t}\right\| \le C \exp\left(-\frac{t}{2}\right).$$