Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 170 (2016) 62–68
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

On one problem of the plane theory of elasticity for a circular domain with a rectangular hole

Gogi Kapanadze[a,b,*], Lida Gogolauri[b]

[a] I. Vekua Institute of Applied Mathematics, I. Javakhishvili Tbilisi State University 2, University st., Tbilisi 0186, GA, United States
[b] A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University 6, Tamarashvili st., Tbilisi 0177, GA, United States

Available online 24 February 2016

Abstract

The paper considers a plane problem of elasticity for a circle with a rectangular hole. To find a solution, the use is made both of the method of conformal mappings and of boundary value problems of analytic functions. In particular, relying on the well-known Kolosov–Muskhelishvili's formulas, the problem formulated with respect to unknown complex potentials is reduced to the two Riemann–Hilbert problems for a circular ring, and the solutions of the latter problems allow us to construct potentials effectively (analytically). The estimates of the obtained results in the neighborhood of angular points are given. Analogous results (as a particular case) are obtained for a circular domain with a rectilinear cut.

Keywords: Conformal mapping; Kolosov–Muskhelishvili's formulas; The Riemann–Hilbert problem for a circular ring

Statement of the Problem. Let S be a doubly-connected domain occupied by a plate on the plane $z = x + iy$ of a complex variable, bounded by circumference $L_0 = \{|z| = R_0\}$ and rectangle $B_1B_2B_3B_4$ whose sides are parallel to the coordinate axes. By L_1 we denote the boundary of the rectangle (that is, $L_1 = \cup_{k=1}^{4} L_k^{(1)}$, $L_k^{(1)} = B_kB_{k+1}$, $k = \overline{1,4}$, $B_5 = B_1$) and assume that the sides B_1B_2 and B_3B_4 (parallel to the ox-axis) are under the action of constant, normal compressive forces with the given principal vector P (or normal displacements $v_n(t) = v_n^{(k)} = \text{const}$, $t \in L_k^{(1)}$, $k = \overline{1,4}$ are given), and the rest of the boundary $L = L_0 \cup L_1$ is free from the external forces.

Note that certain simplifications in the statement of the problem concerning the cut forms and external forces are insignificant and motivated only to make the problem more clear, namely, to find elastic equilibrium of the plate for a finite doubly-connected domain.

* Corresponding author at: I. Vekua Institute of Applied Mathematics, I. Javakhishvili Tbilisi State University 2, University st., Tbilisi 0186, GA, United States.

E-mail addresses: kapanadze.49@mail.ru (G. Kapanadze), lida@rmi.ge (L. Gogolauri).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.02.001

Analogous problems of the plane theory of elasticity and plate bending for finite doubly-connected domains bounded by polygons have been considered in [1,2].

Solution of the Problem. As is known, the more effective ways of solving the boundary value problems of the plane theory of elasticity by the methods of complex analysis are based on the construction of a conformally mapping function of the given domain onto canonical domains (circle, circular ring). Therefore the above-mentioned methods are little-suited for the effective solution of problems in multi-connected domains. Nevertheless, for some practically important classes of multi-connected domains one manages to construct effectively (analytically) the conformally mapping function of that domain onto a circular ring. These classes involve doubly-connected domains bounded by polygons and their modifications (polygonal domain with a circular hole, or a circle with a polygonal hole). Moreover, the Kolosov–Muskhelishvili's methods in the above-mentioned case allow one to decompose these problems (with respect to complex potentials $\varphi(z)$ and $\psi(z)$) into two Riemann–Hilbert problems for a circular ring, and by solving the latter problems to construct unknown potential in analytical form.

Here we present some results (see [3]) dealing with conformal mapping of a doubly-connected domain, bounded by a polygon, onto a circular ring.

(1) The Dirichlet Problem for a Circular Ring. Let $\mathcal{D}(1 < |z| < R)$ be a circular ring bounded by circumferences $\ell_0(|z| = R)$ and $\ell_1(|z| = 1)$. We consider the problem: find a holomorphic in the ring $\mathcal{D}$ function $\varphi_*(z) = u + iv$ under the boundary condition

$$\mathrm{Re}[\varphi_*(t)] = f_j(t), \quad t \in \ell_j, \ j = 0, 1. \tag{1}$$

The necessary and sufficient condition for solvability of problem (1) is of the form

$$\int_0^{2\pi} f_0(t)d\vartheta = \int_0^{2\pi} f_1(t)d\vartheta \tag{2}$$

and a solution itself is given by the formula

$$\varphi_*(z) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}\left[\int_{\ell_0}\frac{f_0(t)}{t - R^{2j}z}\,dt + \int_{\ell_1}\frac{f_1(t)}{t - R^{2j}z}\,dt\right] + ik_1,$$

where k_1 is an arbitrary real constant. Integration on ℓ_0 and ℓ_1 taken in the positive direction leaves the domain D at the left.

(2) Conformal Mapping of a Doubly-Connected Domain, Bounded by Polygons, onto a Circular Ring. Let S^0 be the doubly-connected domain on the plane z of a complex variable, bounded by convex polygons (A) and (B). Assume that (A) is an outer and (B) is an interior boundary of the domain S^0; by A_k $(k = 1, \dots, n)$ and $B_m (m = 1, \dots, p)$ we denote the vertices (and their affixes) and by $L_0^{(k)}$ and $L_1^{(k)}$ the sides of polygons (A) and (B). By $\pi\alpha_k^0$ and $\pi\beta_m^0$ we denote the sizes of inner angles S^0 at the vertices A_k and B_m, and the angles lying between the ox-axis and exterior normals to the contours L_0 $\left(L_0 = \cup_{k=1}^n L_k^{(0)}\right)$ and L_1 $\left(L_1 = \cup_{m=1}^p L_m^{(1)}\right)$ we denote by $\alpha(t)$ and $\beta(t)$; the positive direction on $L = L_0 \cup L_1$ is taken that which leaves the domain S^0 at the left.

Consider the problem: find the type of the function $z = \omega_0(\zeta)$ conformally mapping the circular ring $D(1 < |\zeta| < R)$ onto the domain S_0.

From the equalities

$$t - A_k = i|t - A_k|e^{i\alpha_k(t)}, \quad t \in L_0^{(k)}; \qquad t - B_m = i|t - B_m|e^{i\beta_m(t)}, \ t \in L_1^{(m)},$$

we get

$$\begin{aligned} &\mathrm{Re}\big[t \cdot e^{-i\alpha(t)}\big] = \mathrm{Re}\big[A(t) \cdot e^{-i\alpha(t)}\big], \quad t \in L_0; \\ &\mathrm{Re}\big[t \cdot e^{-i\beta(t)}\big] = \mathrm{Re}\big[B(t) \cdot e^{-i\beta(t)}\big], \quad t \in L_1, \end{aligned} \tag{3}$$

where $A(t)$, $B(t)$, $\alpha(t)$ and $\beta(t)$ are the piecewise constant functions;

$$A(t) = A_k; \qquad \alpha(t) = \alpha_k(t), \quad t \in L_0^{(k)};$$
$$B(t) = B_m, \qquad \beta(t) = \beta_m(t), \quad t \in L_1^{(m)}.$$