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Original article

On negligible and absolutely nonmeasurable subsets of uncountable solvable groups

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Abstract

It is proved that every uncountable solvable group contains two negligible sets whose union is an absolutely nonmeasurable subset of the same group.

Keywords: Solvable group; Invariant measure; Quasi-invariant measure; Negligible set; Absolutely nonmeasurable set

In this paper we will be dealing with measures invariant (or, more generally, quasi-invariant) under various transformation groups. We will be interested in the behavior of certain sets with respect to such measures. The notation and terminology used in the paper is primarily taken from [1] and [2]. All basic facts of modern measure theory can be found in [3]. An extensive survey devoted to measures given on different algebraic-topological structures is presented in [4].

Let E be a base (ground) set and let G be some group of transformations of E. In this case, the pair (E, G) is usually called a space equipped with a transformation group.

We shall say that a set $X \subset E$ is G-negligible (in E) if the following two conditions are fulfilled for X:

(a) there exists at least one nonzero σ-finite G-invariant (G-quasi-invariant) measure μ on E such that $X \in \mathrm{dom}(\mu)$;

(b) for every σ-finite G-invariant (G-quasi-invariant) measure ν on E such that $X \in \mathrm{dom}(\nu)$, the equality $\nu(X) = 0$ holds true.

We shall say that a set $Y \subset E$ is G-absolutely nonmeasurable (in E) if, for any nonzero σ-finite G-quasi-invariant measure θ on E, we have $X \notin \mathrm{dom}(\theta)$.

If $(G, \cdot)$ is a group, then we may consider G as a ground set E and take the group of all left translations of G as a group of transformations of E. Obviously, identifying G with the group of all left translations of G, we may speak of left G-invariant (left G-quasi-invariant) measures on E (=G) and, respectively, we may consider G-negligible and G-absolutely nonmeasurable subsets of G.

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Example 1. If $(G, \cdot)$ is an arbitrary uncountable solvable group, then there exists a G-absolutely nonmeasurable subset of G (in this connection, see e.g. [2] and references therein). At the same time, it is still unknown whether there exists a Γ-absolutely nonmeasurable set in any uncountable group $(\Gamma, \cdot)$.

The main goal of this paper is to show (for a certain class of spaces (E, G)) that there exist two G-negligible sets in E, the union of which turns out to be G-absolutely nonmeasurable in E. In particular, if E itself is an uncountable solvable group and G coincides with the group of all left translations of E, then the above-mentioned fact is valid for (E, G). Clearly, this yields some generalization of the statement formulated in Example 1.

It should be noticed that basic technical tools which lead us to the required result are motivated by the method of surjective homomorphisms (cf. [1,2,5]).

For our further purposes, we need several auxiliary propositions. The first of them is essentially contained in [2].

As usual, the symbol $\omega (=\omega_0)$ denotes the least infinite cardinal (ordinal) number and ω_1 denotes the least uncountable cardinal (ordinal) number.

Lemma 1. *Let a space (E, G) satisfy the following two relations:*

(1) $\mathrm{card}(E) = \omega_1$ *and the group G acts freely and transitively in E;*

(2) *there are two subgroups G_0 and G_1 of G such that*

$$\mathrm{card}(G_0) = \omega, \qquad \mathrm{card}(G_1) = \omega_1, \qquad G_0 \cap G_1 = \{\mathrm{Id}_E\},$$

where Id_E is the identity transformation of E.

Then there exist two G-negligible subsets T_1 and T_2 of E such that the set $T_1 \cup T_2$ is G-absolutely nonmeasurable in E.

Proof. We would like to recall one construction of a G-absolutely nonmeasurable subset of E (see [2], Chapter 11, Lemma 3). First, let us observe that relation (1) directly implies the equality

$$\mathrm{card}(G) = \omega_1.$$

So we may take an ω_1-sequence $\{\Gamma_\xi : \xi < \omega_1\}$ of subgroups of G, such that:

(a) $\Gamma_0 = G_0$;

(b) for all ordinals $\xi < \omega_1$, we have $\mathrm{card}(\Gamma_\xi) = \omega$;

(c) for each ordinal $\xi < \omega_1$, the set $\cup\{\Gamma_\zeta : \zeta < \xi\}$ is a proper subset of Γ_ξ (in particular, this ω_1-sequence of subgroups of G is strictly increasing by inclusion);

(d) $\cup\{\Gamma_\xi : \xi < \omega_1\} = G$.

Further, fix a point $y \in E$ and, for any ordinal number $\xi < \omega_1$, put

$$Y_\xi = \Gamma_\xi(y) \setminus \cup\{\Gamma_\zeta(y) : \zeta < \xi\}.$$

A straightforward verification shows that the family of sets $\{Y_\xi : \xi < \omega_1\}$ forms a partition of E and each Y_ξ is a Γ'_ξ-invariant subset of E, where the group Γ'_ξ is defined by the formula

$$\Gamma'_\xi = \cup\{\Gamma_\zeta : \zeta < \xi\}.$$

According to relation (c), the group Γ'_ξ is a proper subgroup of Γ_ξ. Also, by virtue of the free action of G in E, it is not hard to see that

$$\mathrm{card}(Y_\xi) = \omega \quad (\xi < \omega_1).$$

Now, for each ordinal number $\xi < \omega_1$, introduce the group

$$G_{1,\xi} = G_1 \cap \Gamma'_\xi.$$

Obviously, the ω_1-sequence $\{G_{1,\xi} : \xi < \omega_1\}$ of groups is increasing by inclusion and

$$\cup\{G_{1,\xi} : \xi < \omega_1\} = G_1.$$

Fix for a while an ordinal $\xi < \omega_1$ and consider the two partitions of Y_ξ into orbits associated with the groups G_0 and $G_{1,\xi}$, respectively. Taking into account the free action of G in E and the relation

$$G_0 \cap G_1 = \{\mathrm{Id}_E\},$$