Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 170 (2016) 75–90
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Sharp weighted bounds for multiple integral operators

Vakhtang Kokilashvili[a,b], Alexander Meskhi[a,c,*], Muhammad Asad Zaighum[d,e]

[a] Department of Mathematical Analysis, A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6. Tamarashvili Str., Tbilisi 0177, Georgia
[b] International Black Sea University, 3 Agmashenebeli Ave., Tbilisi 0131, Georgia
[c] Department of Mathematics, Faculty of Informatics and Control Systems, Georgian Technical University, 77, Kostava St., Tbilisi, Georgia
[d] Department of Mathematics and Statistics, Riphah International University, I-14, Islamabad, Pakistan
[e] Pontificia Universidad Javeriana, Departamento de Matemáticas, Cra. 7, Bogotá, Colombia

Available online 13 January 2016

Abstract

Sharp weighted bounds for strong maximal functions, multiple potentials and singular integrals are derived in terms of Muckenhoupt type characteristics of weights.

Keywords: Strong maximal functions; Integral operators with product kernels; Boundedness; One-weight inequality; Sharp bound

1. Introduction

In this paper, we establish sharp weighted bounds for strong maximal functions and multiple integral operators. Our derived results involve, in particular, Buckley-type estimates for strong Hardy–Littlewood and fractional maximal functions, potentials and singular integrals with product kernels, and their one-sided analogs.

One of the main problems in Harmonic Analysis is to characterize a weight w for which a given integral operator is bounded in L^p_w (one-weight inequality). An important class of such weights is the well-known A_p class. It is known that A_p condition is necessary and sufficient for the boundedness of Hardy–Littlewood and singular integral operators (see, e.g., [1–3]); however, the sharp dependence of the corresponding L^p_w norms in terms of A_p characteristic of w is known only for some operators. The interest in the sharp weighted norm, for example, for singular integral operators is motivated by applications in partial differential equations (see e.g., [4–7]).

Strong maximal operator different from the usual one is defined with respect to parallelepipeds with sides parallel to the co-ordinate axes; the operators with product kernels, such as multiple singular and potential operators have singularities not only at a single point but on the hyperplanes. That is why to study mapping properties for such

* Corresponding author at: Department of Mathematical Analysis, A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 6. Tamarashvili Str., Tbilisi 0177, Georgia.
E-mail addresses: kokil@rmi.ge (V. Kokilashvili), meskhi@rmi.ge (A. Meskhi), asadzaighum@gmail.com (M.A. Zaighum).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2015.12.001
2346-8092/

operators became more complicated; however from the one weight viewpoint it is possible to get one-weight boundedness results as well as sharp weighted bounds by deducing the problem to the single variable result and using repeatedly the latter one uniformly with respect to other variables. In this direction Proposition 2.1 is one of the keys to get the main results. One of the important aspects of this paper is that this point enables us to get sharp one-weight results for a quite large class of multiple operators including one-sided cases.

Let X and Y be two Banach spaces. Given a bounded operator $T : X \to Y$, we denote the operator norm by $\|T\|_{X\to Y}$ which is defined in the standard way i.e. $\|T\|_{X\to Y} := \sup_{\|f\|_X\le 1} \|Tf\|_Y$. If $X = Y$ we use the symbol $\|T\|_X$.

An almost everywhere positive locally integrable function (i.e. weight) w defined on $\mathbb{R}^n$ is said to satisfy $A_p(\mathbb{R}^n)$ condition ($w \in A_p(\mathbb{R}^n)$) for $1 < p < \infty$ if

$$\|w\|_{A_p(\mathbb{R}^n)} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where $p' = \frac{p}{p-1}$ and supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides parallel to the co-ordinate axes. We call $\|w\|_{A_p(\mathbb{R}^n)}$ the A_p characteristic of w.

In 1972 B. Muckenhoupt [3] showed that if $w \in A_p(\mathbb{R}^n)$, where $1 < p < \infty$, then the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy$$

is bounded in $L^p_w(\mathbb{R}^n)$.

S. Buckley [8] investigated the sharp A_p bound for the operator M and established the inequality

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le C\|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty. \tag{1.1}$$

Moreover, he showed that the exponent $\frac{1}{p-1}$ is best possible in the sense that we cannot replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here it follows that for any $\lambda > 0$,

$$\sup_{w\in A_p} \frac{\|M\|_{L^p_w}}{\|w\|^{\frac{1}{p-1}-\lambda}_{A_p}} = \infty.$$

To explain better the point of sharp estimates for multiple operators, let us discuss, for example, the strong Hardy–Littlewood maximal operator $M^{(s)}$ defined on $\mathbb{R}^2$. Denote by $A^{(s)}_p(\mathbb{R}^2)$ the Muckenhoupt class taken with respect to the rectangles with sides parallel to the co-ordinate axes (see Section 2 for the definitions). Let $\|w\|_{A^{(s)}_p(\mathbb{R}^2)}$ be $A^{(s)}_p$ characteristic of w. There arises a natural question regarding the sharp bound in the inequality

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^2)} \le c\|w\|^{\beta}_{A^{(s)}_p(\mathbb{R}^2)}. \tag{1.2}$$

We show that the following estimate is sharp

$$\|M^{(s)}\|_{L^p_w(\mathbb{R}^2)} \le c\left(\|w\|_{A_p(x_1)}\|w\|_{A_p(x_2)}\right)^{1/(p-1)}, \tag{1.3}$$

where $\|w\|_{A_p(x_i)}$ is the characteristic of the weight w defined with respect to the ith variable uniformly to another one $i = 1, 2$ (see e.g., [9–11], Ch. IV for the one-weight theory for multiple integral operators). Inequality (1.3) together with the Lebesgue differentiation theorem implies that (1.2) holds for $\beta = \frac{2}{p-1}$; however, unfortunately we do not know whether it is or not sharp.

Under the symbol $A \approx B$ we mean that there are positive constants c_1 and c_2 (depending on appropriate parameters) such that $c_1A \le B \le c_2A$; $A \ll B$ means that there is a positive constant c such that $A \le cB$.