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Filtered Hirsch algebras

Samson Saneblidze

A. Razmadze Mathematical Institute I. Javakhishvili Tbilisi State University 6, Tamarashvili st., Tbilisi 0177, Georgia

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Abstract

Motivated by the cohomology theory of loop spaces, we consider a special class of higher order homotopy commutative differential graded algebras and construct the filtered Hirsch model for such an algebra A. When $x \in H(A)$ with $\mathbb{Z}$ coefficients and $x^2 = 0$, the symmetric Massey products $\langle x\rangle^n$ with $n \geq 3$ have a finite order (whenever defined). However, if $\Bbbk$ is a field of characteristic zero, $\langle x\rangle^n$ is defined and vanishes in $H(A \otimes \Bbbk)$ for all n. If p is an odd prime, the Kraines formula $\langle x\rangle^p = -\beta\mathcal{P}_1(x)$ lifts to $H^*(A \otimes \mathbb{Z}_p)$. Applications of the existence of polynomial generators in the loop homology and the Hochschild cohomology with a G-algebra structure are given.

Keywords: Hirsch algebra; Filtered model; Multiplicative resolution; Symmetric Massey product; Steenrod operation; Hochschild cohomology

1. Introduction

In this paper we investigate a special class of homotopy commutative algebras called *Hirsch algebras* [20]. When the structural operations of a Hirsch algebra A agree component-wise with those of a homotopy G-algebra (HGA), the pre-Jacobi axiom can fail [7,8,19,37] and the induced product on the bar construction BA is not necessarily associative. Indeed, the theory of loop space cohomology suggests that it is impossible in general, to construct a small model for $H^*(\Omega X)$ in the category of HGAs. The investigation here applies a perturbation theory that extends the well-developed perturbation theories for differential graded modules and differential graded algebras (dgas) [3,9,13,11,27,28].

One difficulty encountered when constructing a theory of homological algebra for Hirsch algebras is that the Steenrod cochain product $a \smile_1 b$ fails to be a cocycle even for cocycles a and b. Consequently $a \smile_1 b$ does not necessarily lift to cohomology. We control such difficulties by introducing the notion of a *filtered* Hirsch algebra, which can be thought of as a specialization of a distinguished resolution in the sense of [10] (see also [14]). On the other hand, the filtered Hirsch model $(RH, d + h)$ of a Hirsch algebra A is itself a Hirsch algebra whose structural operations $E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \longrightarrow RH$ are completely determined by the commutative graded algebra (cga) structure of $H = H(A, d_A)$; furthermore, the perturbation $h : RH \to RH$ of the resolution differential d is

E-mail address: sane@rmi.ge.
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determined by the Hirsch algebra structure on A (Theorem 1). Thus by ignoring the operations $E_{p,q}$ we obtain a multiplicative resolution $(RH, d) \to (H, 0)$ of the cga H thought of as a non-commutative version of its Tate–Jozefiak resolution [35,16] and the filtered model of the dga A is the perturbation $(RH, d + h) \to (A, d_A)$ in [27] (such a filtered model in the category of cdgas over a field of characteristic zero was constructed by Halperin and Stasheff in [11]).

A Hirsch resolution always admits a binary operation $\cup_2$, which can be viewed as *divided* Steenrod $\smile_2$-operation. This leads to the notion of a *quasi-homotopy commutative* Hirsch algebra (QHHA) introduced here. We note that in general, the construction of a Hirsch map $(RH, d + h) \to A$ compatible with a QHHA structure on A is obstructed by the non-free action of Sq_1 on its cohomology $H(A)$.

Every cdga H can be thought of as a trivial Hirsch algebra in which the operations $E_{p,q} \equiv 0$ for all $p, q \geq 1$. However, we exhibit an example of a cohomology algebra $H = H(A)$ with a non-trivial Hirsch algebra structure determined by Sq_1.

For a Hirsch algebra A over the integers, we establish some formulas relating the structural operations $E_{p,q}$ with syzygies in (RH, d) that arise from a single element $x \in H(A)$ with $x^2 = 0$. Whereas the n-fold symmetric Massey product $\langle x\rangle^n$ with $n \geq 3$ is defined in $H(A)$ [23,22], our formulas imply that $\langle x\rangle^n$ has finite order. Note that when A is an algebra over a field $\Bbbk$ of characteristic zero, $\langle x\rangle^n$ is defined and vanishes for all $n \geq 3$ (Theorem 2). As a consequence we have (compare [4]):

Theorem A. *Let X be a simply connected space, let $\Bbbk$ be a field of characteristic zero and let $\sigma_* : H_*(\Omega X; \Bbbk) \to H_{*+1}(X; \Bbbk)$ be the suspension map. If $y \notin \operatorname{Ker} \sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ for all $n \geq 2$.*

Given an odd prime p, consider the Hirsch algebra $A \otimes \mathbb{Z}_p$, let $x \in H^{2m+1}(A \otimes \mathbb{Z}_p)$, and let β be the Bockstein operator. We obtain the formula

$$\langle x\rangle^p = -\beta\mathcal{P}_1(x), \tag{1.1}$$

which has the same form as Kraines's formula in [23], however, the cohomology operation $\mathcal{P}_1 : H^{2m+1}(A \otimes \mathbb{Z}_p) \to H^{2mp+1}(A\otimes\mathbb{Z}_p)$ in (1.1) is canonically determined by the iteration of the $\smile_1$-product on $A\otimes\mathbb{Z}_p$ (Theorem 3). Dually, if A is the singular chains on the triple loop space $\Omega^3 X$, we can identify $\mathcal{P}_1$ with the Dyer–Lashof operation (see [22]). In fact the validity of (1.1) in a general algebraic framework is conjectured by May [25, Section 6]. Furthermore, when $X = BF_4$, the classifying space of the exceptional group F_4, we exhibit explicit perturbations in the filtered model of X and recover formula (1.1) in $H^*(X; \mathbb{Z}_3)$.

Although Theorem 1 provides a theoretical model of a Hirsch algebra A endowed with higher order operations $E_{p,q}$, in practice one can construct a small *multiplicative* model for recognizing $H^*(BA)$ as an algebra in which the product is determined only by the binary operation $E_{1,1} = \smile_1$. Thus, a (minimal) multiplicative resolution of $H^*(A)$ endowed with a $\smile_1$-product provides an economical way to calculate the algebra $H^*(BA)$. We apply this technique to the Hochschild cochain complex $A = C^\bullet(P; P)$ of an associative algebra P over a field $\Bbbk$ of characteristic zero to establish the following.

Theorem B. *If the Hochschild cohomology $H^* = H(C^\bullet(P; P))$ is a free algebra, then the Lie algebra structure on $Tor^A_*(\Bbbk, \Bbbk)$ is completely determined by that of the G-algebra H^*. Consequently, the product μ^* on $Tor^A_*(\Bbbk, \Bbbk)$ is commutative if and only if the G-product on H^* is trivial.*

Some applications of filtered Hirsch algebras considered in an earlier version of this paper are also considered in [31,32] (see also [29,33]).

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2. The category of Hirsch algebras

This section defines the generalized notion of a Hirsch algebra applied here, the morphisms between them, and the notion of a Hirsch resolution.

Let $\Bbbk$ be a commutative ring with unity 1 and characteristic ν; in the applications, $\Bbbk$ will be the integers $\mathbb{Z}$, a finite field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ with p prime, or a field of characteristic zero. Graded $\Bbbk$-modules A^* are assumed to be graded over $\mathbb{Z}$. A module A^* is connected if $A^0 = \Bbbk$, and a non-negatively graded, connected module A^* is 1-*reduced* if $A^1 = 0$.