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Explicit expressions for the moments of the size of an $(s, s+1)$-core partition with distinct parts

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For fixed s, the size of an $(s, s+1)$-core partition with distinct parts can be seen as a random variable X_s. Using computer-assisted methods, we derive formulas for the expectation, variance, and higher moments of X_s in terms of s. Our results give good evidence that X_s is asymptotically normal.

1. Introduction: the size of an (s, t)-core partition

Recall that the *hook length* of a box in the Young diagram of a partition is the number of boxes to the right (the arm) plus the number of boxes below it (the leg) plus one (the head). (We use the English convention for Young diagrams; see Fig. 1.) A partition is an *s-core* if its Young diagram avoids hook length s and an (s, t)*-core* if it avoids hook

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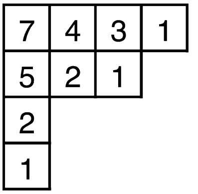

Fig. 1. Young diagram of the partition $9 = 4 + 3 + 1 + 1$, showing the hook lengths of each box.

lengths s and t [2]. For example, the partition $9 = 4 + 3 + 1 + 1$ in Fig. 1 is a $(6, 8)$-core but not a $(6, 7)$-core.

The number of (s, t)-core partitions is finite iff s and t are coprime, which we shall assume from now on [2]. Let $X_{s,t}$ be the random variable "size of an (s, t)-core partition," where the sample space is the set of all (s, t)-core partitions, equipped with the uniform distribution. In [3], with the help of Maple, Zeilberger derived explicit expressions for the expectation, variance, and numerous higher moments of $X_{s,t}$. The original paper noted that "From the 'religious-fanatical' viewpoint of the current 'mainstream' mathematician, they are 'just' conjectures, but nevertheless, they are **absolutely certain** (well, at least as absolutely certain as most proved theorems)," and a donation to the OEIS was offered for the theory to make the results rigorous. Later, it was found that such theory did exist and the results are entirely rigorous; see the updates at the paper's site.

Zeilberger also computed some standardized central moments of $X_{s,t}$ and the limit of these expressions as $s, t \to \infty$ with $s - t$ fixed. From this he conjectured the limiting distribution. Perhaps surprisingly, it is abnormal.

Here, we continue the experimental approach taken up in [3]. However, we consider (s, t)-core partitions with *distinct* parts. Further, we make the restriction $t = s+1$. Using Maple, we are able to again conjecture, and in this case *rigorously prove* the validity of, explicit expressions for the moments in terms of s. Further, we show that the limiting distribution *does* appear normal in this case.

2. Distinct part $(s, s + 1)$-core partitions

2.1. Computing the generating function

Given a positive integer s, let P_s be the set of all $(s, s+1)$-core partitions with distinct parts. Observe that $|P_s|$ is always finite. Let X_s be the random variable "size of a partition in P_s." Our goal is to have an efficient way to compute the generating function

$$G_s(q) := \sum_{p \in P_s} q^{|p|}$$

for fixed s. (Here $|p|$ denotes the size of a partition p.) This will then allow us to compute moments of X_s.