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Elliptic extensions of the alpha-parameter model and the rook model for matchings ☆

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ABSTRACT

We construct elliptic extensions of the alpha-parameter rook model introduced by Goldman and Haglund and of the rook model for matchings of Haglund and Remmel. In particular, we extend the product formulas of these models to the elliptic setting. By specializing the parameter α in our elliptic extension of the alpha-parameter model and the shape of the Ferrers board in different ways, we obtain elliptic analogues of the Stirling numbers of the first kind and of the Abel polynomials, and an a, q-analogue of the matching numbers. We further generalize the rook theory model for matchings by introducing $\mathbf{l}$-lazy graphs which correspond to $\mathbf{l}$-shifted boards, where $\mathbf{l}$ is a finite vector of positive integers. The corresponding elliptic product formula generalizes Haglund and Remmel's product formula for matchings already in the non-elliptic basic case.

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1. Introduction

Since the introduction of rook theory by Kaplansky and Riordan [12], the theory has thrived and developed further by revealing connections to, for instance, orthogonal polynomials [7,9], hypergeometric series [10], q-analogues and permutation statistics [4,5], algebraic geometry [2,3], and many more. Within rook theory itself, various models have been introduced, including a p, q-analogue of rook numbers [1,14,20], the j-attacking model [14], the matching model [11], the augmented rook model [13] which includes all other models as special cases, etc. In previous work [17,18], the authors have constructed elliptic extensions of the aforementioned rook theory models and obtained corresponding product formulas.

In this work, we construct elliptic extensions of two rook theory models: the alpha parameter rook model introduced by Goldman and Haglund [8] and the matching model of Haglund and Remmel [11]. The alpha-parameter model, as explained in [8], is a slight generalization of the i-creation model. It connects to several other combinatorial models, including polynomial sequences of binomial type, permutations of multisets, Abel polynomials, Bessel polynomials and matchings, and so on. Our elliptic extension lays the foundations for raising those connections to the elliptic level.

In our construction of an elliptic analogue of the matching model, we actually consider a model that generalizes the original model of Haglund and Remmel already in the non-elliptic, basic case. In particular, we consider matchings on specific graphs which we call "$\mathbf{l}$-lazy graphs" with respect to an N-dimensional vector of positive integers, $\mathbf{l} = (l_1, l_2, \ldots, l_N)$. The original matching model can be realized from the generalized model by setting $N = 2n - 1$ and $\mathbf{l} = (1, 1, \ldots, 1)$. For the new model, we are able to prove a product formula involving elliptic rook numbers for matchings on $\mathbf{l}$-lazy graphs, a result which generalizes the corresponding product formula of Haglund and Remmel [11].

In Section 2 we define elliptic weights and review some of the elementary identities useful for dealing with them. Section 3 is devoted to the elliptic extension of the alpha-parameter model, together with some applications. Finally, Section 4 features an elliptic extension of the rook theory of matchings.

2. Elliptic weights

In this section, we define the elliptic weights which we utilize to weight cells in Ferrers boards. (For the definition of Ferrers boards, see Section 3.) We start by explaining what elliptic functions are.

A complex function is called elliptic, if it is a doubly-periodic, meromorphic function on $\mathbb{C}$. It is well-known that such functions can be expressed in terms of ratios of theta functions (cf. [21]). We will use the following (multiplicative) notation for theta functions. First, we define the *modified Jacobi theta function* with argument x and nome p by